

From Preferences to Judgments and Back

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Abstract

The paper studies the interrelationships of the Preference Aggregation and Judgment Aggregation problems from the point of view of logical semantics. The result of the paper is twofold. On the one hand, the Preference Aggregation problem is viewed as a special case of the Judgment Aggregation one. On the other hand, the Judgment Aggregation problem is viewed as a special case of the Preference Aggregation one. It is shown how to import an impossibility result from each framework to the other.

1 Introduction

Recent results [5, 9] have shown how Preference Aggregation theorems, such as Arrow's impossibility [1], can be obtained as corollaries of impossibility theorems concerning the aggregation of judgments. The idea behind this reduction consists in viewing preferences between issues as special kind of judgments, i.e., formulae to which a truth-value is attached. In [5, 9] the formulae used for representing preferences are first-order formulae of the type xPy (" x is strictly preferred to y ") where x, y are variables for the elements in the set of issues of the Preference Aggregation problem. Then, in order for the judgments concerning such formulae to behave like a strict preference relation the three axioms of asymmetry, transitivity and connectedness¹ are added to the Judgment Aggregation framework.

The present paper proposes a different approach to obtain the same kind of reduction. More precisely, preferences will be studied as implicative statements $y \rightarrow x$ (" y is at most as preferred as x ") in a many-value logic setting [6, 7]. The insights gained by this reduction of preferences to judgments are then also used to obtain an inverse reduction of judgments to special kind of preferences, namely those preferences definable in the Boolean algebra on the support $\{0, 1\}$. To the best of our knowledge, this is the first work advancing a proposal on how to reduce Judgment Aggregation problems to Preference Aggregation ones.

The paper is structured as follows. In Section 2 the frameworks of Preference and Judgment Aggregation are briefly exposed, some basic terminology is introduced and some relevant results from the literature are summarized. In Section 3 a reduction of preferences to judgments is proposed which makes use of the semantics of many-valued logics. An impossibility result from Judgment Aggregation is thereby imported into Preference Aggregation. Section 4 explains how the framework of Preference Aggregation could be extended in order to incorporate preferences between complex issues representable as logical formulae. In Section 5 such idea is related to propositional logic and a reduction of judgments to preferences is proposed. Also in this case, an impossibility result of Preference Aggregation is imported into Judgment Aggregation. Section 6 briefly concludes.

2 Preliminaries

The present section is devoted to the introduction of the two frameworks of preference and judgment aggregation, and of some results which will be of use later in the paper.

¹Asymmetry: $\forall x, y(xPy \rightarrow \neg yPx)$; Transitivity: $\forall x, y, z((xPy \wedge yPz) \rightarrow xPz)$; Connectedness: $\forall x, y(x \neq y \rightarrow (xPy \vee yPx))$

2.1 Preference Aggregation

Preference Aggregation (PA) concerns the aggregation of the preferences of several agents into one collective preference. A preference relation \preceq on a set of issues Iss^P is a total preorder, i.e., a binary relation which is reflexive, transitive, and total. $\mathfrak{T}(\text{Iss}^P)$ denotes the set of all total preorders of a set Iss^P . As usual, on the ground of \preceq we can define its asymmetric and symmetric parts: $x \prec y$ iff $(x, y) \in \preceq$ & $(y, x) \notin \preceq$; $x \approx y$ iff $(x, y) \in \preceq$ & $(y, x) \in \preceq$. We also define from \preceq the following non-transitive relation: $x \lesssim y$ iff $(x, y) \in \preceq$ or $(y, x) \in \preceq$ but not both. The notion of PA structure can now be defined.

Definition 1. (*Preference aggregation structure*) A PA structure is a quadruple $\mathfrak{S}^P = \langle \text{Agn}^P, \text{Iss}^P, \text{Prf}^P, \text{Agg}^P \rangle$ where: Agn^P is a finite set of agents such that $1 \leq |\text{Agn}^P|$; Iss^P is a finite set of issues such that $3 \leq |\text{Iss}^P|$; Prf^P is the set of all preference profiles, i.e., $|\text{Agn}^P|$ -tuples $\mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}$ where each \preceq_i is a total preorder over Iss ; Agg^P is a function taking each $\mathbf{p} \in \text{Prf}^P$ to a total preorder over Iss , i.e., $\text{Agg}^P : \text{Prf}^P \longrightarrow \mathfrak{T}(\text{Iss}^P)$. The value of Agg^P is denoted by \preceq .

The aggregation function Agg^P is then studied under the assumption that it satisfies some of the following typical conditions:

Unanimity (**U**): $\forall x, y \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}$ if $\forall i \in \text{Agn}^P, y \prec_i x$ then $y \prec x$.

Independence (**I**): $\forall x, y \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}, \mathbf{p}' = (\preceq'_i)_{i \in \text{Agn}^P}$, if $\forall i \in \text{Agn}^P y \preceq_i x$ iff $y \preceq'_i x$, then $y \preceq x$ iff $y \preceq' x$.

Systematicity (**Sys**) $\forall x, y, x', y' \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}, \mathbf{p}' = (\preceq'_i)_{i \in \text{Agn}^P}$, if $\forall i \in \text{Agn}^P y \preceq_i x$ iff $y' \preceq'_i x'$, then $y \preceq x$ iff $y' \preceq' x'$.

Non-dictatorship (**NoDict**): $\nexists i \in \text{Agn}^P$ such that $\forall x, y \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}$, if $y \prec_i x$ then $y \prec x$.

Notice that Definition 1 incorporates also the aggregation conditions usually referred to as universal domain and collective rationality.

2.2 Judgment Aggregation

Judgment aggregation (JA) concerns the aggregation of sets of interrelated formulae into one collective set of formulae. This section introduces the framework for judgment aggregation based on the language of propositional logic [9].

A central notion in Judgment Aggregation is the notion of agenda. Intuitively, an agenda consists of all the possible positions that agents can assume about the truth and falsity of some issue.

Definition 2. (*JA agenda*) The language of propositional logic is denoted by \mathcal{L} . Given a finite set of formulae $\text{Iss}^J \subseteq \mathcal{L}$, the set $\text{ag}(\text{Iss}^J) = \{ \models \phi \mid \phi \in \text{Iss}^J \} \cup \{ \not\models \phi \mid \phi \in \text{Iss}^J \}$ denotes the agenda of Iss^J .

A JA agenda consists therefore of a set of pairs $(\models \phi, \not\models \phi)$, each member in the pairs meaning that ϕ is assigned value 1 and, respectively, a different value from 1, which is in the propositional case 0. It might be worth noticing that we assume a slightly different perspective on agendas than the literature on JA. Normally, an agenda is viewed as a set of position/negation pairs $(\phi, \neg\phi)$. In this view, the judgment themselves can be seen as formulae of the language from which the issues are drawn. Instead, we prefer to see judgments as meta-formulae stating whether a given issue is accepted (true) or rejected (not true). In propositional logic, however, the two perspectives are equivalent.

A judgment set picks one element out of each such pairs keeping propositional consistency. More formally, a *judgment set* for the agenda $\mathbf{ag}(\mathbf{Iss}^J)$ is a set $J \subseteq \mathbf{ag}(\mathbf{Iss}^J)$ which is consistent and complete ($\forall \phi \in \mathbf{Iss}^J$: either $\models \phi \in J$ or $\not\models \phi \in J$ but not both) and closed under propositional logic consequence ($\forall \phi, \psi \in \mathbf{Iss}^J$: if $\phi \models \psi$ and $\models \phi \in J$ then $\models \psi \in J$). The set of all judgment sets for the agenda built on \mathbf{Iss}^J is denoted by $\mathfrak{J}(\mathbf{Iss}^J)$. The set \mathbf{Iss}_0^J denotes the set of propositional atoms in \mathbf{Iss}^J .

Definition 3. (*Judgment aggregation structure*) A judgment aggregation structure is a quadruple $\mathfrak{S}^J = \langle \mathbf{Agn}^J, \mathbf{Iss}^J, \mathbf{Prf}^J, \mathbf{Agg}^J \rangle$ where: \mathbf{Agn}^J is a finite set of agents such that $1 \leq |\mathbf{Agn}^J|$; \mathbf{Iss}^J is a finite set of issues consisting of propositional formulae and containing at least two atoms: $\mathbf{Iss}^J \subseteq \mathcal{L}$ s.t. $2 \leq |\mathbf{Iss}_0^J|$; \mathbf{Prf}^J is the set of all judgment profiles, i.e., $|\mathbf{Agn}^J|$ -tuples $\mathbf{j} = \{J_i\}_{i \in \mathbf{Agn}^J}$ where each J_i is a judgment set for the agenda $\mathbf{ja}(\mathbf{Iss}^J)$; \mathbf{Agg}^J is a function taking each $\mathbf{j} \in \mathbf{Prf}^J$ to a judgment set for $\mathbf{ag}(\mathbf{Iss}^J)$, i.e., $\mathbf{Agg}^J : \mathbf{Prf}^J \longrightarrow \mathfrak{J}(\mathbf{Iss}^J)$. The value of \mathbf{Agg}^J is denoted by J .

Like in PA, in JA aggregation functions are studied under specific conditions. The following conditions reformulate the ones proper of PA:

Unanimity (**U**): $\forall x \in \mathbf{Iss}^J$ and $\forall \mathbf{j} = (J_i)_{i \in \mathbf{Agn}^J}$ if $\forall i \in \mathbf{Agn}^J, \models x \in J_i$ then $\models x \in J$ and if $\not\models x \in J_i$ then $\not\models x \in J$.

Independence (**I**): $\forall x \in \mathbf{Iss}^J$ and $\forall \mathbf{j} = (J_i)_{i \in \mathbf{Agn}^J}, \mathbf{j}' = (J'_i)_{i \in \mathbf{Agn}^J}$, if $\forall i \in \mathbf{Agn}^J \models x \in J_i$ iff $\models x \in J'_i$ then $\models x \in J$ iff $\models x \in J'$.

Systematicity (**Sys**): $\forall x, y \in \mathbf{Iss}^J$ and $\forall \mathbf{j} = (J_i)_{i \in \mathbf{Agn}^J}, \mathbf{j}' = (J'_i)_{i \in \mathbf{Agn}^J}$, if $\forall i \in \mathbf{Agn}^J \models x \in J_i$ iff $\models y \in J'_i$, then $\models x \in J$ iff $\models y \in J'$, and if $\models x \in J_i$ iff $\not\models y \in J'_i$ then $\models x \in J$ iff $\not\models y \in J'$.

Non-dictatorship (**NoDict**): $\nexists i \in \mathbf{Agn}^J$ such that $\forall x \in \mathbf{Iss}^J$ and $\forall \mathbf{j} = (J_i)_{i \in \mathbf{Agn}^J}$, if $\models x \in J_i$ then $\models x \in J$ and if $\not\models x \in J_i$ then $\not\models x \in J$.

Notice that Definition 3 incorporates the aggregation conditions usually referred to as universal domain and collective rationality. In the remainder of the paper we will refer to the conditions of unanimity, independence, systematicity and non-dictatorship as **U**, **I**, **Sys**, respectively, **NoDict**. It will be clear from the context whether the condition at issue should be interpreted in its PA or in its JA formulation.

2.3 Some relevant results on JA and PA

We now briefly sketch two results which are of importance for the development of the work presented in this paper: Propositions 1 and 2.

JA agendas can be studied from the point of view of their structural properties, that is, how strictly connected are the issues with which the agenda is concerned. In [5] the following structural property —among others— is studied.

Definition 4. (*Minimal connectedness of the agenda*) An agenda $\mathbf{ag}(\mathbf{Iss}^J)$ is minimally connected if: i) it has a minimal inconsistent subset $S \subseteq \mathbf{ag}(\mathbf{Iss}^J)$ such that $3 \leq |S|$; ii) it has a minimal inconsistent subset $S \subseteq \mathbf{ag}(\mathbf{Iss}^J)$ such that $(S \setminus Z) \cup \{\not\models x \mid \models x \in Z\}$ is consistent for some $Z \subseteq \mathbf{ag}(\mathbf{Iss}^J)$ of even size.

A typical JA impossibility result making use of this property is the following [5, 10]. In Section 3 this result will then be imported, as an example, from JA to PA.

Proposition 1. (*Impossibility for minimally connected agendas*) For any JA structure \mathfrak{S}^J there exists no aggregation function for a minimally connected agenda $\mathbf{ag}(\mathbf{Iss}^J)$ which satisfies **U**, **Sys** and **NoDict**.

| $\{x, y\}$ | $\{y, z\}$ | $\{x, z\}$ | $\{x, y\}$ | $\{y, z\}$ | $\{x, z\}$ |
|-------------|-------------|-------------|---------------|---------------|---------------|
| $y \prec x$ | $z \prec y$ | $z \prec x$ | $u(y) < u(x)$ | $u(z) < u(y)$ | $u(z) < u(x)$ |
| $y \prec x$ | $y \prec z$ | $x \prec z$ | $u(y) < u(x)$ | $u(y) < u(z)$ | $u(x) < u(z)$ |
| $x \prec y$ | $z \prec y$ | $x \prec z$ | $u(x) < u(y)$ | $u(z) < u(y)$ | $u(x) < u(z)$ |
| $y \prec x$ | $z \prec y$ | $x \prec z$ | $u(y) < u(x)$ | $u(z) < u(y)$ | $u(x) < u(z)$ |

Table 1: Condorcet’s paradox.

Proof. We refer the reader to [5]. □

A ranking function attributes a ranking (or value, or utility, or payoff) to all the issues of a preference aggregation problem. In this paper we will make use of ranking functions with the real interval $[0, 1]$ as codomain. For such functions the following result holds which is the special case of a theorem first proven in [4]. This result will play a central role in the next section.

Proposition 2. (*Representation of \preceq*) *Let \preceq be a total preorder on a finite set X . There exists a ranking function $u : X \rightarrow [0, 1]$ such that $\forall x, y \in X : x \preceq y$ iff $u(x) \leq u(y)$. Such a function is unique up to ordinal transformations².*

Proof. The reader is referred to [4]. □

3 Preferences as judgments

This section is devoted to show how the aggregation of preferences can be studied in terms of the aggregation of judgments. As anticipated in Section 1 we get to the very same conclusions presented in [5]. Nevertheless, to obtain such result we will follow a different approach based on logical semantics rather than logical axiomatics. Such approach will offer, in Section 5, also a method for viewing judgments as forms of preferences.

3.1 Condorcet’s paradox as a judgment aggregation paradox

In Condorcet’s paradox, pairwise majority voting on issues generates a collective preference which is not transitive. From Proposition 2 we know that any preference relation which is a total preorder can be represented by an appropriate ranking function u with codomain $[0, 1]$. Table 1 depicts the standard version of the paradox in relational notation, and the version using ranking functions u . To obtain the paradox strict preferences are not necessary. The paradox arises also in weaker forms like the one depicted in Table 2. Again, both the relational and the ranking function-based versions are provided.

The basic intuition underlying this section consists in reading the right-hand sides of Tables 1 and 2 as if u was an interpretation function of propositions x, y, z on the real interval $[0, 1]$. In many-valued logic [6, 7], a semantic clause such as $u(y) \leq u(x)$ typically defines the satisfaction by u of the implication $y \rightarrow x$:

$$u \models y \rightarrow x \text{ iff } u(y) \leq u(x). \tag{1}$$

Intuitively, implication $y \rightarrow x$ is true (or accepted, or satisfied) iff the rank of y is at most as high as the rank of x . Since we know by Proposition 2 that any total preorder \preceq can

²We recall that an ordinal transformation t is a function such that for all utilities m and n , $t(m) \leq t(n)$ iff $m \leq n$.

| $\{x, y\}$ | $\{y, z\}$ | $\{x, z\}$ | $\{x, y\}$ | $\{y, z\}$ | $\{x, z\}$ |
|---------------|---------------|---------------|------------------|------------------|------------------|
| $y \preceq x$ | $z \preceq y$ | $z \preceq x$ | $u(y) \leq u(x)$ | $u(z) \leq u(y)$ | $u(z) \leq u(x)$ |
| $y \preceq x$ | $y \prec z$ | $x \prec z$ | $u(y) \leq u(x)$ | $u(y) < u(z)$ | $u(x) < u(z)$ |
| $x \prec y$ | $z \preceq y$ | $x \prec z$ | $u(x) < u(y)$ | $u(z) \leq u(y)$ | $u(x) < u(z)$ |
| $y \preceq x$ | $z \preceq y$ | $x \prec z$ | $u(y) \leq u(x)$ | $u(z) \leq u(y)$ | $u(x) < u(z)$ |

Table 2: Weak Condorcet's paradox.

be represented by a ranking function, the bridge between the notion of preference and an equivalent notion of judgment is thereby readily available: given a total preorder \preceq , there always exists a ranking function u , unique up to order-preserving transformations such that: $y \preceq x$ iff $u(y) \leq u(x)$. We thus obtain a direct bridge between preferences and judgments. In fact, by exploiting Formula 1 the right-hand side of Table 2 can be rewritten as in Table 3. Notice that x is substituted by p , y by q and z by r . The same could obviously be done for Table 1.

In other words, by first reading the weak Condorcet's paradox in terms of ranking function (Proposition 2), and then interpreting such functions from the point of view of logical semantics (Formula 1), we can show the equivalence between a preference aggregation problem and a judgment aggregation one. The following result generalizes this observation.

Proposition 3. *(From preferences to judgments) The set of all PA structures can be mapped into the set of all JA structures in such a way that each of them corresponds exactly to one JA structure whose set of issues Iss^J consists of only implications.*

Proof. We show how to construct a structure \mathfrak{S}^J from any structure \mathfrak{S}^P . Let $\mathfrak{S}^P = \langle \text{Agn}^P, \text{Iss}^P, \text{Prf}^P, \text{Agg}^P \rangle$. The set of agents Agn^J of \mathfrak{S}^J is the same: $\text{Agn}^P = \text{Agn}^J$. The set of issues Iss^J is such that: $\text{Iss}_0^J = \text{Iss}^P$, that is, Iss^P provides the propositional atoms of Iss^J ; and it contains all implications built from Iss_0^J . The set Prf^J is the set of all judgment profiles obtained by translating any total order \preceq_i into a judgment set J_i as follows: $\models x \rightarrow y \in J_i$ iff $(x, y) \in \preceq_i$. Notice that, as a consequence, we can define a bijection bi between Prf^P and Prf^J . Finally, the aggregation function Agg^J can be defined as follows: $\text{Agg}^J(\text{bi}(\mathbf{p})) = \text{bi}(\text{Agg}^P(\mathbf{p}))$. This completes the construction. \square

The JA structures \mathfrak{S}^J resulting from the construction in the proof are such that their set of issues Iss^J consist of all implications obtained from a set of atoms Iss_0^J such that $3 \leq |\text{Iss}_0^J|$. We call an agenda $\text{ag}(\text{Iss}^J)$ built on such a set Iss^J an *implicative agenda*.

| $\{p, q\}$ | $\{q, r\}$ | $\{p, r\}$ | $q \rightarrow p$ | $r \rightarrow q$ | $r \rightarrow p$ |
|-------------------------------|-------------------------------|-------------------------------|-------------------|-------------------|-------------------|
| $\models q \rightarrow p$ | $\models r \rightarrow q$ | $\models r \rightarrow p$ | \models | \models | \models |
| $\models q \rightarrow p$ | $\not\models r \rightarrow q$ | $\not\models r \rightarrow p$ | \models | $\not\models$ | $\not\models$ |
| $\not\models q \rightarrow p$ | $\models r \rightarrow q$ | $\not\models r \rightarrow p$ | $\not\models$ | \models | $\not\models$ |
| $\models q \rightarrow p$ | $\models r \rightarrow q$ | $\models r \rightarrow p$ | \models | \models | $\not\models$ |

Table 3: Weak Condorcet's paradox as a judgment aggregation paradox.

3.2 Doing PA in JA

We know how to translate a PA structure into a JA one with implicative agendas. As an example of how to import impossibility results of JA to PA we apply Proposition 1 to JA structures with implicative agendas. In order to do so, we first need to show that implicative agendas enjoy the property of minimal connectedness.

Proposition 4. *Implicative agendas are minimally connected.*

Proof. Suppose $\{p, q, r\} \subseteq \text{Iss}_0^J$. We prove that the implicative agenda built on Iss_0^J is minimally connected. The desired minimal inconsistent subset of the agenda with size higher than or equal to 3, and such that by negating two of its judgments consistency is restored, is: $\{\models q \rightarrow p, \models r \rightarrow q, \not\models r \rightarrow p\}$. \square

Everything is in place now to prove an impossibility result concerning the set of JA structures in which PA can be mapped, i.e., those with implicative agendas.

Theorem 1. *(A JA theorem for PA) For any PA structure \mathfrak{S}^P , there exists no aggregation function which satisfies **U**, **Sys** and **NoDict**.*

Proof. The theorem follows from Propositions 1, 3 and 4. \square

The theorem itself is, of course, not surprising. It is just an illustration of the embedding advanced in this section. More interesting results can be obtained along the very same lines by studying different structural properties of the agenda, e.g., strong connectedness [5].

Before closing the present section it is worth spending a few words on the relation between the approach presented here and the one presented in [5] where Arrow’s theorem is proven as a corollary of a JA theorem analogous to Proposition 1 and a “bridging” proposition analogous to Proposition 4. Both approaches somehow reduce PA to JA, but while [5] does it axiomatically by imposing further constraints on a first-order logic agenda (i.e., the axioms of strict total orders), we do it semantically, by ranking the issues in Iss^P on the $[0, 1]$ interval and interpreting preferences as implications. A more in-depth comparison of the two approaches deserves further investigation. However, the semantic view has the advantage of hinting also a “way back” from judgments to preferences. Such “way back” is the topic of the next two sections.

4 Intermezzo: Rankings as Truth Values

The essential difference between JA and PA is that, in JA, issues display logical form. Is there a consistent way to talk about compound issues in PA obtained by performing logical operations on atomic ones? In other words, is there a way to define preferences which display the logical complexity of judgments? Aim of the present section is to show how to answer these questions by generalizing Formula 1 to any logical connective.

Once we consider the set of issues Iss^P of a preference aggregation problem \mathfrak{S}^P to be the finite set of propositional atoms \mathcal{L}_0 of a propositional language \mathcal{L} , any ranking function u can be viewed as an interpretation function of the atoms in \mathcal{L}_0 on the real interval $[0, 1]$. The natural question follows: how to inductively extend a function u in order to interpret issues in Iss^P which consist of propositional formulae, and not just atoms? This question takes us into the realm of many-valued logics, and many possibilities are available. However, given Formula 1, we are looking for something in particular. We want an implication to be satisfied exactly when the antecedent is ranked at most as high as the consequent. More precisely, let us denote with \tilde{u} the inductive extension of the ranking function u . What we

are looking for is a multi-valued logic such that the following holds for any ranking function u and formulae ϕ, ψ :

$$u \models \phi \rightarrow \psi \quad \text{iff} \quad \check{u}(\phi) \leq \check{u}(\psi) \quad \text{iff} \quad \check{u}(\phi \rightarrow \psi) = 1. \quad (2)$$

That is to say, the desired logic should be able to encode in the language the total order \leq of rankings, so that \check{u} assigns the maximum ranking 1 to $\phi \rightarrow \psi$ (i.e., $\phi \rightarrow \psi$ is satisfied by u) iff the value assigned by \check{u} to ϕ is at most the same value assigned by \check{u} to ψ . The intuition behind Formula 2 consists in viewing the maximum ranking as the designated value for expressing the truth of compound formulae, and in particular of those formulae which express preferences between other formulae.

The property expressed in Formula 2 turns out to be a typical property of the family of t-norm multi-valued logics, or logics based on triangular norms [7]. In such logics the connective \rightarrow denotes the algebraic residuum operation on truth degrees (i.e., rankings). Residua come always in pairs with t-norm operations, so what is going to characterize the logic we are looking for is the t-norm we choose to be paired with the residuum denoted by \rightarrow . The most straightforward candidate is the algebraic infimum, denoted by the standard logic conjunction \wedge . To sum up, we want that the following holds for any ranking function u and formulae ϕ, ψ, ξ :

$$\check{u}(\phi \wedge \xi) \leq \check{u}(\psi) \quad \text{iff} \quad \check{u}(\xi) \leq \check{u}(\phi \rightarrow \psi). \quad (3)$$

If we then take the rest of the connectives \neg and \vee to denote, as usual, the algebraic complementation and, respectively, the algebraic supremum, the many-valued logic satisfying Formulae 2 and 3 is the logic known as Gödel-Dummett logic (**GD** in short).

By using the semantics of **GD** (i.e., Gödel algebra) it is possible to extend the PA framework in order to incorporate preferences between compound issues represented as logical formulae. What kind of new insights this extension provides in the study of the aggregation of preferences deserves further research, but it falls outside the scope of the present study. In fact, what we are interested in now is to show that the insights provided by **GD** on the representation of preferences between logical formulae can be used, after a slight modification, in order to view judgments as preferences, thereby providing a reduction of the JA problem to the PA problem. To this aim is devoted the next section.

5 Judgments as Preferences

In its original form, JA is based on propositional logic. The considerations of the last section about how to interpret compound issues in a PA structure can be directly used for representing judgments as preferences. The key step consists in considering that propositional logic is the extension of **GD** with the bivalence principle. Like an interpretation function of **GD** (i.e., a ranking function u) determines a total preorder on the set of issues of a PA structure, so does a propositional interpretation function on the set of issues of a JA structure. Obviously, the type of total preorder yielded by a propositional interpretation function is of a specific kind.

5.1 Boolean preference profiles

Like **GD** is sound and complete w.r.t. Gödel algebra [7], propositional logic is sound and complete w.r.t. $\mathbf{2} = \langle \{0, 1\}, \sqcap, \sqcup, -, 0, 1 \rangle$, i.e., the Boolean algebra on the support $\{1, 0\}$, where \sqcap and \sqcup are the **min**, respectively, **max** operations, $-$ is the involution defined as $-x = 1 - x$, and 0 and 1 are the designated elements. The total preorders generated by a propositional interpretation function are called *Boolean preferences*.

Definition 5. (*Boolean preferences*) A Boolean preference is a total preorder on a set of formulae Φ closed under atoms which can be mapped to $\langle \{1, 0\}, \leq \rangle$ by a function $\check{v} : \Phi \rightarrow \{1, 0\}$ such that: $\forall \phi, \psi \in \text{Iss}^J$, $\phi \preceq \psi$ iff $\check{v}(\phi) \leq \check{v}(\psi)$; and \check{v} preserves the meaning of propositional connectives, that is:

$$\check{v}(\top) = 1, \quad \check{v}(\neg\phi) = 1 - \check{v}(\phi), \quad \check{v}(\phi \wedge \psi) = \min(\check{v}(\phi), \check{v}(\psi)), \quad \check{v}(\phi \vee \psi) = \max(\check{v}(\phi), \check{v}(\psi)).$$

A few considerations are in order. Notice that, within a Boolean preference, \prec -paths have maximum length 1. In fact, stating that $y \prec x$ is equivalent to assign value 1 to x and value 0 to y . Notice also that a total preorder containing $x \prec y, y \prec x \wedge y$ cannot be a Boolean preference since there exists no function assigning 1 and 0 to x and y , which preserves \preceq on \leq and, at the same time, \wedge on \min . Notice also that Boolean preferences generalize dichotomous preferences³ allowing issues to be compounded following the standard algebraic semantics of logical connectives, and allowing issues to be all ranked as maximal or minimal. In fact, dichotomous preferences are nothing but Boolean preferences over atoms, i.e., logically unrelated issues, s.t. the sets of maximal and minimal elements always contain at least one atom (i.e., the preferences are never “unconcerned” [3]). In addition, Boolean preferences feature the constants \top and \perp . These denote trivial issues which every voter places in the sets of \preceq -maximal and, respectively, \preceq -minimal elements. The following holds.

Proposition 5. (*From judgment sets to Boolean preferences*) Every judgment set J on the set of issues Iss^J can be translated to a Boolean preference \preceq over Iss^J such that $\models \phi \in J$ iff $\phi \approx \top$.

Proof. Since each judgment set J is closed under atoms, it univocally determines a propositional evaluation $\check{v} : \text{Iss}^J \rightarrow \{1, 0\}$. The evaluation is an homomorphism from the formula algebra built on the set of atoms in Iss^J and **2**. In **2** the partial order \leq can be defined in the usual way: $x \leq y := \min(x, y) = x$. It is easy to see that \leq is then a total preorder. Function \check{v} of J can therefore be used to univocally define a total preorder \preceq on Iss^J as follows: $\forall \phi, \psi \in \text{Iss}^J$, $\phi \preceq \psi$ iff $\check{v}(\phi) \leq \check{v}(\psi)$. It follows that \preceq is a Boolean preference by construction. From left to right. If $\models \phi \in J$ then $\check{v}(\phi) = 1$, therefore, by construction $\phi \approx \top$. From right to left. If $\phi \approx \top$ then $\check{v}(\phi) = \check{v}(\top)$, hence $\check{v}(\phi) = 1$ \square

Intuitively, Proposition 5 guarantees the analogue of Proposition 3 to hold. In other words, every JA structure \mathfrak{G}^J can be translated to an equivalent PA structure⁴ by just stating $\text{Iss}^P = \text{Iss}^J$ and letting Prf^P be the set of Boolean preference profiles on Iss^P . We thus obtain a way to view judgments as preferences. As an example, in the next section, we will consider the Discursive paradox from a PA point of view.

5.2 The Discursive paradox as a PA paradox

In the Discursive paradox propositionwise majority voting leads to the definition of an impossible evaluation of the propositions at issue, as showed in the left-hand side of Table 4. The middle of Table 4 depicts the Discursive paradox by means of the two truth-values. Finally, if we consider these truth values as rankings of the propositions at issue, we get to the right-hand side of the table. There a PA version of the paradox is depicted. Recall that $x \approx y$ iff $(x, y) \in \preceq$ & $(y, x) \in \preceq$, and $x \leq y$ iff $(x, y) \in \preceq$ or $(y, x) \in \preceq$ but not both. In this case the aggregated preference violates the transitivity of \approx .

As such, the Discursive paradox can fruitfully be viewed as yet another variant of Condorcet’s paradox.⁵ In fact, in Condorcet’s paradox the collective preference violates the

³Dichotomous preferences are such that $\forall x, y, z \in \text{Iss}^P$, either $x \approx y$ or $y \approx z$ or $x \approx z$ but not all [8].

⁴Notice that the same does not hold for dichotomous preferences where we cannot distinguish between preferences ranking all issues equal to \top or all equal to \perp (see also Proposition 7).

⁵Other representations are possible by making use of the constants \top or \perp , and indifference \approx .

| p | $p \rightarrow q$ | q | p | $p \rightarrow q$ | q | $\{p, p \rightarrow q\}$ | $\{p, q\}$ | $\{q, p \rightarrow q\}$ |
|---------------|-------------------|---------------|-----|-------------------|-----|-----------------------------|---------------|-----------------------------|
| \models | \models | \models | 1 | 1 | 1 | $p \approx p \rightarrow q$ | $p \approx q$ | $q \approx p \rightarrow q$ |
| \models | $\not\models$ | $\not\models$ | 1 | 0 | 0 | $p \leq p \rightarrow q$ | $q \leq p$ | $q \approx p \rightarrow q$ |
| $\not\models$ | \models | $\not\models$ | 0 | 1 | 0 | $p \leq p \rightarrow q$ | $p \approx q$ | $q \leq p \rightarrow q$ |
| \models | \models | $\not\models$ | 1 | 1 | 0 | $p \leq p \rightarrow q$ | $p \approx q$ | $q \approx p \rightarrow q$ |

Table 4: Discursive paradox as a PA paradox.

transitivity of \prec , in the weak Condorcet analyzed in Section 3 what is violated is instead the transitivity of \preceq , here it is the transitivity of \approx .

5.3 Arrow's conditions and Boolean preferences

We can now represent judgments as special kinds of preferences. Can we also import PA impossibility results to JA? A first natural step in this direction is to study how Boolean preferences behave under the standard Arrow's conditions: **U**, **I** and **NoDict**. Under these conditions, and assuming the domain and codomain of the aggregation function to be drawn from the set of Boolean preferences, yields an impossibility. However, the source of the impossibility does not rest upon the logical connectives, as is typically the case in JA.

Proposition 6. *(Arrow's theorem holds for dichotomous domains and codomains) Let \mathfrak{S}^P contain a set of issues Iss^P s.t. $3 \leq |\text{Iss}^P|$, and Prf^P is the set of dichotomous preference profiles on Iss^P . There exists no aggregation function which satisfies **U**, **I** and **NoDict**.*

Proof. See Appendix. \square

Limiting the domain and codomain of the aggregation function to dichotomous preferences does not resolve the impossibility. Since dichotomous preferences are a subset of Boolean preferences the result carries over. Conditions **U**, **I** and **NoDict** seem therefore to be too strong to yield JA-like impossibilities. This is not surprising because, unlike in the translation from PA to JA (Section 3), the JA formulation of the conditions cannot be directly obtained from their PA formulation. The translation of the standard JA conditions in a Boolean preferences format look instead like this:

(**U**[†]): $\forall x \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}$ if $\forall i \in \text{Agn}^P, x \approx_i \top$ then $x \approx \top$ and if $x \approx_i \perp$ then $x \approx \perp$.

(**I**[†]): $\forall x \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}, \mathbf{p}' = (\preceq'_i)_{i \in \text{Agn}^P}$, if $\forall i \in \text{Agn}^P x \approx_i \top$ iff $x \approx'_i \top$ then $x \approx \top$ iff $x \approx' \top$.

(**Sys**[†]): $\forall x, y \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}, \mathbf{p}' = (\preceq'_i)_{i \in \text{Agn}^P}$, if $\forall i \in \text{Agn}^J x \approx_i \top$ iff $y \approx'_i \top$, then $x \approx \top$ iff $y \approx' \top$, and if $x \approx_i \top$ iff $y \approx'_i \perp$, then $x \approx \top$ iff $y \approx' \perp$.

(**NoDict**[†]): $\nexists i \in \text{Agn}^P$ such that $\forall x \in \text{Iss}^P$ and $\forall \mathbf{p} = (\preceq_i)_{i \in \text{Agn}^P}$, if $x \approx_i \top$ then $x \approx \top$ and if $x \approx_i \perp$ then $x \approx \perp$.

It is straightforward to see that these conditions exactly correspond to the standard JA conditions exposed in Section 2. The following proposition compares the relative strength of these JA-like formulations w.r.t. the standard PA-like ones.

Proposition 7. *(JA vs. PA conditions) The following relations hold under Boolean preferences: **U**[†] implies **U**; **I**[†] implies **I**; **Sys**[†] implies **Sys**; **NoDict** implies **NoDict**[†]. The converses do not hold.*

Proof. The direction from left to right is trivial. The failure of the converses is due to the fact that for profiles in which all agents are indifferent w.r.t. all issues, it cannot be inferred whether they are all ranked equal to \perp or equal to \top . \square

Notice that in the case of non-dictatorship the implication has a contrapositive form. In fact, **NoDict** is stronger than **NoDict** $^\top$. That is, the non-existence of a dictator in the standard Arrowian sense implies the non-existence of a dictator in the JA sense. This observation suggests that results such as Proposition 6 set out to prove impossibilities on the ground of a notion of non-dictatorship which is stronger than the one typically used in JA. The next section fine-tunes Proposition 6 obtaining an appropriate JA impossibility in the framework of Boolean preferences.

5.4 Doing JA in PA

In order to obtain an impossibility for Boolean preferences under **NoDict** $^\top$, the conditions **U** $^\top$ and **I** $^\top$ do not suffice and, perhaps not surprisingly, **I** $^\top$ needs to be substituted by **Sys** $^\top$. We are thus in the position to prove a JA theorem within PA.

Theorem 2. (*Impossibility for Boolean Preferences*) *Let \mathfrak{S}^P contain a set of issues Iss^P s.t. $\{p, q, p \wedge q\}$ (where \wedge can be substituted by \vee or \rightarrow) and \mathbf{Prf}^P is the set of Boolean preference profiles on Iss^P . There exists no aggregation function which satisfies **U** $^\top$, **Sys** $^\top$ and **NoDict** $^\top$.*

Proof. See Appendix. \square

Now, Proposition 5 guarantees that each JA structure \mathfrak{S}^J can be translated to an equivalent PA structure \mathfrak{S}^P over Boolean preferences, and the standard JA aggregation conditions can be directly translated to conditions **U** $^\top$, **Sys** $^\top$ and **NoDict** $^\top$. It follows that Theorem 2 provides an impossibility result for the aggregation of judgments. Notice also that such result is reminiscent of several JA theorems available in the literature (e.g. in [9]). To conclude, the basic argument of the section runs as follows: judgment sets univocally determine Boolean preferences, a peculiar form of Arrow's impossibility holds also for Boolean preference domains, hence it can be imported into JA.

6 Conclusions

By borrowing ideas from logical semantics, the paper has shown that, on the one hand, PA can be viewed as a special case of JA [5, 9] and that, on the other hand, the converse also holds. This suggests that PA and JA could be studied as the two faces of a same coin. It is our claim that the study of such interrelationship can be fruitfully pushed further by cross importing more (im)possibility results, which we plan to do in future researches.

Acknowledgments

The author is very grateful to Gabriella Pigozzi, Leon van der Torre, Paul Harrenstein and Franz Dietrich. Their comments have greatly improved the present version of the paper.

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A Proofs of Proposition 6 and Theorem 2

A proof can be obtained along the same lines of Arrow’s original proof [2]. Let us first introduce some terminology. A set $V \subseteq \mathbf{Agn}^P$ is *almost decisive* for issue x over issue y (in symbols, $AD_V(x, y)$) iff: if $\forall i \in V, y \prec_i x$ and $\forall i \notin V, x \prec_i y$ then $y \prec x$. A set $V \subseteq \mathbf{Agn}^P$ is *decisive* for issue x over issue y (in symbols, $D_V(x, y)$) iff: if $\forall i \in V, y \prec_i x$ then $y \prec x$. Obviously, for any $x, y \in \mathbf{Iss}^P$: $D_V(x, y)$ implies $AD_V(x, y)$. We first need two following lemmata.

Lemma 1. *Let \mathfrak{S}^P be such that \mathbf{Iss}^P is a set of issues s.t. $3 \leq |\mathbf{Iss}^P|$, and \mathbf{Prf}^P is the set of dichotomous preferences on \mathbf{Iss}^P . If there exists an individual $i \in \mathbf{Agn}^P$ such that $AD_V(x, y)$ for some pair (x, y) then, under the conditions **U** and **I**, i is decisive for any pair of issues, that is, i is a dictator.*

Proof. There are 6 pairs of issues with respect to which i can be almost decisive for the first element in the pair over the second one. For each of these pairs (x, y) we show that if $AD_i(x, y)$ then i is decisive for at least one of the remaining pairs $(x, z), (z, x), (y, z), (z, y)$ where z is the third issue in \mathbf{Iss}^P . Let I denote $\mathbf{Agn}^P - \{i\}$. [$AD_i(x, y) \Rightarrow D(x, z)$] Assume $AD_i(x, y)$ and suppose the following holds: $y \prec_i x, z \approx_i y, z \prec_i x$ and $x \prec_I y$. In the collective preference, which is dichotomous, one of the following must hold: A) $y \prec x, y \prec z$; or B) $y \prec x, z \approx y$ However, A) violates **U** and **I**. To prove this latter claim, consider a profile which is identical to the given one w.r.t. the preferences of i and I over y, z , but such that $x \prec_i y$ in the preference profile we assumed. By **U**, in the collective preference for this new profile $x \prec y$ holds. By **I**, $y \prec z$ should also hold, which is impossible since the collective preference should be dichotomous. Therefore, B) is the only possible collective preference. Since B) is dichotomous, it holds that $z \prec x$, hence $AD_i(x, y)$ implies $D(x, z)$. We can argue symmetrically to prove that $AD_i(x, y) \Rightarrow D(z, y)$. Hence, $AD_i(x, y)$ implies $D(x, z)$ and $D(z, y)$. It is then sufficient to argue via permutations of the possible alternatives,

e.g., by interchanging y with z we obtain that $AD_i(x, z)$ implies $D(x, y)$ and $D(y, z)$. As a consequence, if $AD_i(x, y)$ agent i results to be decisive for any pair of alternatives drawn from the set $\{x, y, z\}$. This completes the proof of the lemma. \square

Lemma 2. *Let \mathfrak{S}^P be such that $I\text{ss}^P$ is a set of issues s.t. $3 \leq |I\text{ss}^P|$, and Prf^P is the set of dichotomous preference profiles on $I\text{ss}^P$. There exists an agent $i \in \text{Agn}^P$ such that i is almost decisive for some pair of issues.*

Proof. Let us proceed per absurdum assuming that there is no almost decisive agent. For condition \mathbf{U} , there always exists for each pair of issues a set which is decisive for that pair, that is, Agn^P . As a consequence, there always exists an almost decisive set for that pair. Let us take the smallest (possibly not unique) decisive set and call such set V . Given our hypothesis, such set cannot be a singleton. Let us divide V in three parts: a singleton $\{i\}$, a set $J = V - \{i\}$, and a set $K = I\text{ss}^P - V$. Suppose the following holds: $y \prec_i x, z \approx_i y, z \prec_i x$; $y \prec_J x, y \prec_J z, z \approx_J x$; and $x \prec_K y, z \approx_K y, z \prec_K x$. As to the collective preference, since V is almost decisive one of the following must hold A) $y \prec x, y \prec z$; or B) $y \prec x, z \approx y$. However, if A) holds then J would be almost decisive contradicting the assumption that V was the smallest such set. It follows that B) is the only option. However, since B) is dichotomous it would also hold that $z \prec x$ which would make i almost decisive for x over z , against our assumption. We can argue symmetrically in the other cases. \square

Proposition 6 follows directly from Lemmata 1 and 2.

As to Theorem 2 we just sketch the proof for lack of space. The same technique of the previous proof can be used by just redefining the notions of decisive and almost decisive sets of agents w.r.t. to single issues. A set $V \subseteq \text{Agn}^P$ is *almost decisive* for formula ϕ (in symbols, $AD_V(\phi)$) iff: if $\forall i \in V, \phi \approx_i \perp$ (respectively, \top) and $\forall i \notin V, \phi \approx_i \top$ (respectively, \perp) then $\phi \approx \top$ (respectively, \perp). A set $V \subseteq \text{Agn}^P$ is *decisive* for ϕ (in symbols, $D_V(\phi)$) iff: if $\forall i \in V, \phi \approx_i \top$ (respectively, \perp) then $\phi \approx_i \top$ (respectively, \perp). Obviously, for any $\phi \in I\text{ss}^P$: $D_V(\phi)$ implies $AD_V(\phi)$. We need again two lemmata.

Lemma 3. *Let \mathfrak{S}^P contain a set of issues $I\text{ss}^P$ s.t. $\{p, q, p \wedge q\}$, and Prf^P is the set of Boolean preferences on $I\text{ss}^P$. If there exists an individual $i \in \text{Agn}^P$ such that $AD_V(\phi)$ for some pair ϕ then, under the conditions \mathbf{U}^\top and \mathbf{Sys}^\top , i is decisive for any formula, that is, i is a dictator.*

Proof. We show that if i is almost decisive for any of the propositions in $\{p, q, p \wedge q\}$ then it is decisive for all of them. Let I denote $\text{Agn}^P - \{i\}$. [$AD_i(p) \Rightarrow D(p \wedge q), D(q)$] The proof proceeds like for Lemma 1 but makes use of systematicity. Assume $AD_i(p)$ and suppose the following holds: $p \approx_i \top, q \approx_i \top, p \wedge q \approx_i \top$ and $p \approx_I \perp$. In the collective preference, which must be Boolean, one of the following must hold: A) $p \approx \top, q \approx \top$; or B) $p \approx \top, q \approx \perp$. However, B cannot be the case because of \mathbf{U}^\top and \mathbf{Sys}^\top . In fact, if $q \approx \perp$ then $q \approx_I \perp$ otherwise, for \mathbf{U}^\top , we should have $q \approx \top$. Consider a profile identical w.r.t. to q and $p \wedge q$ but s.t. $p \approx_i \perp$. Then by \mathbf{U}^\top it follows that $p \approx \perp$, and by \mathbf{Sys}^\top it follows that $q \approx \top$ since the two profiles are such that p is ranked \top in the first iff q is ranked \top in the second, which is impossible. Hence, A) is the only option, which proves the claim. We can then argue symmetrically for the other cases. \square

The analogous of Lemma 2 can easily be proven, which completes the proof of the theorem.

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