Towards a Dichotomy for the Possible Winner Problem in Elections Based on Scoring Rules

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joint work with

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Typical voting scenario for joint decision making:

Voters give preferences over a set of candidates as linear orders.

Example: candidates: $C = \{a, b, c, d\}$

profile: vote 1: $a > b > c > d$
vote 2: $a > d > c > b$
vote 3: $b > d > c > a$

Aggregate preferences according to a voting rule

Kind of voting rules considered in this work: **Scoring rules**
Preferences as linear orders, scoring rules. Reminder:

Examples:

- plurality: $(1, 0, \ldots, 0)$
- 2-approval: $(1, 1, 0, \ldots, 0)$
- veto: $(1, \ldots, 1, 0)$
- Borda: $(m - 1, m - 2, \ldots, 0)$ ($m =$ number of candidates)
- Formula 1 scoring: $(25, 18, 15, 12, 10, 8, 6, 4, 2, 1, 0, \ldots, 0)$
Scoring rules

$m$ candidates: scoring vector $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ with \(\alpha_1 \geq \alpha_2 \geq \cdots \geq \alpha_m\) and \(\alpha_m = 0\)

Scoring rule

provides a scoring vector for every number of candidates.

- non-trivial: \(\alpha_1 \neq 0\)
- pure: the scoring vector for \(i\) candidates can be obtained from the scoring vector for \(i-1\) candidates by inserting an additional score value at an arbitrary position

Example:

3 candidates: \((6, 3, 0)\)

4 candidates: pure: \((6, 3, 3, 0), (6, 5, 3, 0), (8, 6, 3, 0), \ldots\)
not pure: \((6, 6, 0, 0), (6, 3, 2, 1), \ldots\)
Partial information

Recall: In the typical model, votes need to be presented as linear orders.

Realistic settings: voters may only provide partial information. For example:
- not all voters have given their preferences yet
- new candidates are introduced
- a voter cannot compare several candidates because of lack of information/because he doesn’t want to

How to deal with partial information?

We consider the question if a distinguished candidate can still win.
A **partial vote** is a transitive and antisymmetric relation.

Example: $C = \{a, b, c, d\}$

partial vote: $a \succ b \succ c, a \succ d$

possible **extensions**:

1. $a > d > b > c$
2. $a > b > d > c$
3. $a > b > c > d$

An extension of a profile of partial votes extends every partial vote.
Computational Problem

**Possible Winner**

**Input:** A voting rule \( r \), a set of candidates \( C \), a profile of partial votes, and a distinguished candidate \( c \).

**Question:** Is there an extension profile where \( c \) wins according to \( r \)?
Known results for scoring rules

Two studied scenarios for **POSSIBLE WINNER**:

1. **weighted voters:**
   NP-completeness for all scoring rules except plurality (holds even for a constant number of candidates)
   (follows by dichotomy for the special case of **MANIPULATION** [Hemaspaandra and Hemaspaandra, JCSS 2007])

2. **unweighted voters:**
   a) constant number of candidates: always polynomial time
      [Conitzer, Sandholm, and Lang, JACM 2007]
   b) unbounded number of candidates:
Known results for scoring rules

- unweighted voters
  b) unbounded number of candidates:
    - NP-complete for scoring rules that fulfill the following:
      - [Xia and Conitzer, AAAI 2008]
      - there is a position $b$ with
        \[
        \alpha_b - \alpha_{b+1} = \alpha_{b+1} - \alpha_{b+2} = \alpha_{b+2} - \alpha_{b+3}
        \]
        and
        \[
        \alpha_{b+3} > \alpha_{b+4}
        \]
      - Examples: $(\ldots, 6, 5, 4, 3, 0, \ldots)$, $(\ldots, 17, 14, 11, 8, 7, \ldots)$
    - Parameterized complexity study for some scoring rules:
      - [Betzler, Hemmann, and Niedermeier, IJCAI 2009]
      - $k$-approval is NP-hard for two partial votes when $k$ is part of the input
# Main Theorem

## Theorem

For non-trivial pure scoring rules, **Possible Winner** is

- polynomial-time solvable for plurality and veto,
- open for \((2, 1, \ldots, 1, 0)\), and
- NP-complete for all other cases.

Recently, the case \((2, 1, \ldots, 1, 0)\) has been shown to be NP-complete as well! \[\text{Baumeister, Rothe, 2010}\]

Examples for new results:

- 2-approval: \((1, 1, 0, \ldots)\)
- voting systems in which one can specify a small group of favorites and a small group of disliked candidates, like \((2, 2, 2, 1, \ldots, 1, 0, 0)\) or \((3, 1, \ldots, 1, 0)\)
Plurality

Example: $C = \{a, b, c, d\}$, distinguished candidate $c$

$v_1 : a \succ c \succ d, b \succ c$
$v_2 : c \succ a \succ b$
$v_3 : a \succ d \succ b$
$v_4 : a \succ b \succ c$
$v_5 : a \succ c, b \succ d$
Plurality

Example: \( C = \{a, b, c, d\} \), distinguished candidate \( c \)

\( \nu_1 : a \succ c \succ d, b \succ c \)

\( \nu_2 : c \succ a \succ b \quad \Rightarrow \quad c > a > b > d \)

\( \nu_3 : a \succ d \succ b \quad \Rightarrow \quad c > a > d > b \)

\( \nu_4 : a \succ b \succ c \)

\( \nu_5 : a \succ c, b \succ d \)

**Step I:** Maximize score of \( c \)
Plurality

Example: $C = \{a, b, c, d\}$, distinguished candidate $c$

$v_1 : a \succ c \succ d, b \succ c$
$v_2 : c \succ a \succ b$ \quad \Rightarrow \quad c > a > b > d$
$v_3 : a \succ d \succ b$ \quad \Rightarrow \quad c > a > d > b$
$v_4 : a \succ b \succ c$
$v_5 : a \succ c, b \succ d$

**Step I:** Maximize score of $c$

**Step II:** Network flow
Example: $C = \{a, b, c, d\}$, distinguished candidate $c$

$v_1 : a \succ c \succ d, b \succ c \quad \Rightarrow a > b > c > d$
$v_2 : c \succ a \succ b \quad \Rightarrow c > a > b > d$
$v_3 : a \succ d \succ b \quad \Rightarrow c > a > d > b$
$v_4 : a \succ b \succ c \quad \Rightarrow d > a > b > c$
$v_5 : a \succ c, b \succ d \quad \Rightarrow b > a > c > d$

**Step I:** Maximize score of $c$

**Step II:** Network flow
What about non-pure scoring rules?

**Theorem**

For non-trivial pure scoring rules, **Possible Winner** is

- polynomial-time solvable for plurality and veto,
- open for $(2, 1, \ldots, 1, 0)$, and
- NP-complete for all other cases.

Problem: scoring rules which have “easy” scoring vectors for nearly all number of candidates and still “hard” scoring vectors for some unbounded numbers of candidates

Property of pure scoring rules: can never go back to an easy vector

Examples: $(1, 0, 0), (1, 1, 0, 0) \rightarrow$ not $(1, 0, 0, 0, 0)$ or $(1, 1, 1, 1, 0)$

$(1, 1, 1, 0), (2, 1, 1, 1, 0), \ldots$
Open questions

- How to compare candidates in partial votes? Counting version: In how many extensions does a distinguished candidate win?
- NP-complete problems: Find approximation/exact exponential algorithm
- Parameter number of candidates: combinatorial algorithm?