The Approximability of Dodgson Elections

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Voting rules

- Input: Agents’ preferences (preference profile)

- Output: Winner(s) of the election or a ranking of the alternatives
Condorcet criterion

- Alternative x beats y in a pairwise election if the majority of agents prefers x to y
- Alternative x is a Condorcet winner if x beats any other alternative in a pairwise election
- Condorcet paradox: A Condorcet winner may not exist
- Choose an alternative as close as possible to a Condorcet winner according to some proximity measure
  - Dodgson’s rule
Condorcet paradox

- a beats b
- b beats c
- c beats a
Dodgson’s method

• Dodgson score of x:
  – the minimum number of exchanges between adjacent alternatives needed to make x a Condorcet winner

• Dodgson ranking:
  – the alternatives are ranked in non-decreasing order of their Dodgson score

• Dodgson winner:
  – an alternative with the minimum Dodgson score
An example of Dodgson

Agent 1

Agent 2

Agent 3

Agent 4

Agent 5

\[ P(a,b) \]
\[ P(a,c) \]
\[ P(a,d) \]
\[ P(a,e) \]
Related combinatorial problems

• Dodgson score:
  – Given a preference profile, a particular alternative x, and an integer K, is the Dodgson score of x at most K?
  – NP-complete: Bartholdi, Tovey, and Trick (Social Choice & Welfare, 1989)

• Dodgson winner:
  – Given a preference profile and a particular alternative x, is x a Dodgson winner?
  – NP-hard: Bartholdi, Tovey, and Trick (Social Choice & Welfare, 1989) and Hemaspaandra, Hemaspaandra, and Rothe (J. ACM, 1997)

• Dodgson ranking:
  – Given a preference profile, compute a Dodgson ranking
Approximation algorithms

• Can we approximate the Dodgson score and ranking?
• i.e., is there an algorithm which, on input a preference profile and a particular alternative \( x \), computes a score which is at most a multiplicative factor away the Dodgson score of \( x \)?
• A \( \rho \)-approximation algorithm guarantees that Dodgson score of \( x \) ≤ score returned by the algorithm for \( x \) ≤ \( \rho \) times Dodgson score of \( x \)
• An approximation algorithm naturally defines an alternative voting rule
Our results

• Approximation of Dodgson’s rule
  – Greedy algorithm
  – An algorithm based on linear programming

• Inapproximability results for the Dodgson ranking and Dodgson score
The greedy algorithm

• Input:
  – A preference profile and a specific alternative x

• Notions:
  – \( \text{def}(x,c) = \) number of additional agents that must rank x above c in order for x to beat c in a pairwise election
  – c is alive iff \( \text{def}(x,c) > 0 \), otherwise dead
  – Cost-effectiveness of pushing alternative x upwards at the preference of an agent = ratio between the number of alive alternatives overtaken by x over number of positions pushed

• Greedy algorithm: While there are alive alternatives, perform the most cost-effective push
The greedy algorithm: an example
Greedy algorithm performance

• **Theorem**: The greedy algorithm has approximation ratio at most $H_{m-1}$

• The proof uses the equivalent ILP for the computation of Dodgson score and its LP relaxation as analysis tools
ILP for Dodgson score

• Variables $y_{ij}$:
  – 1 if agent $i$ pushes $x$ $j$ positions, 0 otherwise

• Constants $a^c_{ij}$:
  – 1 if pushing $x$ $j$ positions in agent $i$ gives $x$ an additional vote against $c$, 0 otherwise

\[
\begin{align*}
\text{minimize} & \quad \sum_{i,j} j \cdot y_{ij} \\
\text{subject to:} & \quad \sum_{j} y_{ij} = 1 \quad \forall i \\
& \quad \sum_{i,j} a^c_{ij} \cdot y_{ij} \geq \text{def}(x,c) \quad \forall c \neq x \\
& \quad y_{ij} \in \{0,1\} \quad \forall i, j
\end{align*}
\]
LP relaxation for Dodgson score

• Variables $y_{ij}$ are fractional
• Constants $a_{ij}^c$:
  – 1 if pushing $x$ to position $j$ in agent $i$ gives $x$ an additional vote against $c$, 0 otherwise

minimize \( \sum_{i,j} j \cdot y_{ij} \)
subject to :
\( \sum_j y_{ij} = 1 \quad \forall i \)
\( \sum_{i,j} a_{ij}^c \cdot y_{ij} \geq \text{def}(x, c) \quad \forall c \neq x \)
\( 0 \leq y_{ij} \leq 1 \quad \forall i, j \)
Greedy algorithm performance

- **Theorem:** The greedy algorithm has approximation ratio at most $H_{m-1}$
- The proof uses the equivalent ILP for the computation of Dodgson score and its LP relaxation as analysis tools
  - We know that $\text{LP} \leq \text{ILP} = \text{Dodgson score}$
  - We use a technique known as dual fitting to show that the score computed by the algorithm is upper bounded by the solution of LP times $H_{m-1}$
  - This means that the greedy algorithm approximates the Dodgson score within $H_{m-1}$
An LP-based algorithm

- Solve the LP and multiply its solution by $H_{m-1}$
- **Theorem:** The LP-based algorithm computes an $H_{m-1}$ – approximation of the Dodgson score
- **Why?**
  - We know that $\text{LP} \leq \text{Dodgson score} \leq \text{LP} H_{m-1}$
  - Hence, $\text{Dodgson score} \leq \text{LP} H_{m-1} \leq \text{Dodgson score times } H_{m-1}$
Inapproximability of Dodgson’s ranking

• **Theorem**: It is NP-hard to decide whether a given alternative is a Dodgson winner or in the last $6 \sqrt{m}$ positions in the Dodgson ranking
  • The proof uses a reduction from vertex cover in 3-regular graphs
• Complexity-theoretic explanation of sharp discrepancies observed in the Social Choice literature when comparing Dodgson voting rule to other (polynomial-time computable) voting rules (e.g., Copeland or Borda)
  • Klamer (Math. Social Sciences, 2004)
  • Ratliff (Economic Theory, 2002)
Inapproximability of Dodgson’s score

• **Theorem:** No polynomial-time algorithm can approximate the Dodgson score of a particular alternative within \((1/2-\varepsilon)\ln m\) unless problems in NP have superpolynomial-time algorithms
  – The proof uses a reduction from Set Cover
A socially desirable property

• A voting rule is weakly monotonic if pushing an alternative upwards in the preferences of some agents cannot worsen its score
• Greedy is not weakly monotonic
• The LP-based algorithm is weakly monotonic
More socially desirable approximations for Dodgson

• In the forthcoming paper:
  – Caragiannis, Kaklamanis, K, & Procaccia (EC 10)
Thank you!