The Weak Axiom of Revealed Preference for Collective Households

Bart Smeulders, Laurens Cherchye, Bram De Rock, Frits C.R. Spieksma, Fabrice Talla Nobibon

Abstract

The purpose of this paper is to derive extensions of the weak axiom of revealed preference (WARP) to collective households. We consider three different settings: the general setting, where no information on the characteristics of goods is available; the private setting, where all goods are consumed privately without externalities, and finally, the dictator setting, where each observation is fully assigned to a single member of the household. For each of the above settings, we propose an extension of WARP and we establish the complexity of testing that extension.

1. Introduction

Modeling and analyzing household consumption behavior is a fundamental research topic in microeconomics since its introduction by Samuelson [14] for *unitary* households, i.e., households consisting of a single member. During the last decades, there has been an increasing interest in analyzing consumption behavior of *collective* households (households consisting of more than one member); and researchers have been focussing on extending well-known revealed preference axioms that were initially developed for unitary households [1, 7, 4, 3]. To date, extensions of the *generalized axiom of revealed preference* (GARP) and of the *strong axiom of revealed preference* (SARP) have received extensive analysis. Results concerning the complexity of testing these axioms indicate that although testing GARP and SARP on data sets of unitary households can be done in polynomial time [20, 12, 18], testing their extensions to collective households is NP-complete [16, 9, 19], even for households with two members.

It is our goal here to consider different extensions of WARP, and to establish the computational complexity of these particular extensions. We aim to complete the picture of testing the collective generalizations of well-known axioms of revealed preference. One observation from practice is that most data that satisfy WARP also satisfy SARP; in other words, transitivity plays little role in the rejection of revealed preference axioms. Therefore, extensions of WARP to collective households provide good alternatives to current procedures for testing collective models that are time-consuming.

We consider collective households that operate in an economy with m commodities or goods that can be consumed either privately without externalities or publicly. *Private* consumption of a good means that its consumption by one household member affects the supply available for other members; as an example, drinking water can only be consumed privately. Consumption *externalities* refer to the fact that one member gets utility from another member's consumption; as an illustration, consider a wife enjoying her husbands nice clothes. *Public* consumption of a good means that consumption of that good by one member does not affect the supply available for other members, and no one can be excluded from consuming it; as example, a movie watched by all members of the household is a public good.

We suppose that we have observed n household consumption quantity bundles $q_t := (q_{t,1}, \ldots, q_{t,m}) \in \mathbb{R}^m_+$ (non-negative) with corresponding positive prices

$p_t := (p_{t,1}, \ldots, p_{t,m}) \in \mathbb{R}^m_{++}$, $t = 1, \ldots, n$. The component $q_{t,i}$ (respectively $p_{t,i}$), for $i = 1, \ldots, m$, corresponds to the quantity of good i bought by the household (respectively, the unit price of good i) at the time of observation t. Note that the scalar product $p'q$ represents the total price of bundle $q \in \mathbb{R}^m_+$ at the prices $p \in \mathbb{R}^m_{++}$. We denote the set of observations by $S := \{(p_t, q_t) : t \in N\}$, where $N := \{1, \ldots, n\}$ and we refer to S as the *data set*. For ease of exposition, throughout this paper, we use $t \in N$ to refer to the observation (p_t, q_t).

We now introduce well-known terminology from revealed preference theory for unitary (single decision maker) households. Throughout this work, we assume that the data set $S := \{(p_t, q_t) : t \in N\}$ represents the consumption of the household. We have the following definitions of preferences [20, 21].

Definition 1 *The decision maker reveals that he or she* directly prefers *bundle q_s over bundle q_t if and only if $p'_s q_s \geq p'_s q_t$.*

In words, the decision maker reveals that he or she directly prefers bundle q_s over bundle q_t if he or she has chosen bundle q_s while bundle q_t was affordable and could have been chosen. When this happens, we simply say that the decision maker directly prefers s over t. Considering the transitive closure of the direct preference relation leads to the next definition.

Definition 2 *The decision maker* prefers *s over t if there exists a sequence $s_1, s_2, \ldots, s_k \in N$, with $s = s_1$ and $t = s_k$, such that he or she directly prefers s_i over s_{i+1}, for $i = 1, 2, \ldots, k-1$.*

These notions are used in the definition of the following well-known axioms of revealed preference, for households with a single decision maker (see, e.g., Varian [20]).

Definition 3 (GARP) *A consumption data set $S := \{(p_t, q_t) : t \in N\}$ satisfies the* Generalized Axiom of Revealed Preference (GARP) *if and only if, for all observations s and t, when the decision maker prefers s over t, then $p'_t q_t \leq p'_t q_s$.*

Definition 4 (SARP) *A consumption data set $S := \{(p_t, q_t) : t \in N\}$ satisfies the* Strong Axiom of Revealed Preference (SARP) *if and only if, for all observations s and t, when $q_s \neq q_t$ and the decision maker prefers s over t, then $p'_t q_t < p'_t q_s$.*

Observe that if S satisfies SARP then it also satisfies GARP. One may consider that the decision maker has a "rational" consumption behavior if S satisfies one of these axioms.

Definition 5 (WARP) *A consumption data set $S := \{(p_t, q_t) : t \in N\}$ satisfies the* Weak Axiom of Revealed Preference (WARP) *if and only if, for all observations s and t, when $q_s \neq q_t$ and the decision maker directly prefers s over t, then $p'_t q_t < p'_t q_s$.*

It is well-known that if S satisfies SARP then it satisfies WARP. Testing whether S satisfies GARP, SARP, or WARP, can be done in time $O(n^2)$ (see [20, 12, 11, 18]); testing whether S satisfies approximate versions of GARP, SARP, or WARP can become more difficult (see [15]).

Review of nonparametric collective models

Collective consumption models explicitly recognize that in a collective household, each individual member has its own preferences. The first such models were introduced by Chiappori [7, 8]. Here, we focus on the nonparametric characterization of the general collective model as presented by Cherchye et al. [3, 5].

Cherchye et al. [3] propose separate necessary and sufficient conditions for collective rationalization of household behaviour. In a later work, they show that the necessary conditions are also sufficient in the case of convex preferences, an observation formalized in the so-called *collective axiom of revealed preference* (CARP) [4]. Because the goal is to identify individual preferences without knowing intra-household consumptions, Cherchye et al. use the term "hypothetical preference". CARP imposes that the hypothetical preference of each individual member satisfies GARP. Talla Nobibon and Spieksma [19] prove that testing CARP is NP-complete, even for two-member households. Cherchye et al. [2] propose an exact algorithm for testing CARP based on a mixed-integer programming formulation. Due to the complexity of the problem, this exact procedure is effective only for small- and medium-size instances. Heuristics for testing large-size instances of this problem are proposed and tested by Talla Nobibon et al. [17]. Cherchye et al. [6] present a variant of the general collective model that applies to the setting where the set of goods is partitioned into privately consumed goods (without externalities) and publicly consumed goods. Talla Nobibon et al. [16] prove that testing this variant of the collective model is also an NP-complete problem. Sabbe [13] present an exact testing procedure based on a mixed-integer programming formulation of the problem. Cherchye et al. [3] describe the general collective model in the dictator setting. Deb [10] shows that this variant of the problem is also NP-complete.

Contributions

The purpose of this paper is to extend WARP to households that consist of two or more decision makers, i.e., *collective* households. We consider the following settings: (i) a *general* setting, where no information on the characteristics of goods is available (Section 2), (ii) the *private* setting where all goods are consumed privately without externalities (Section 3), and (iii) the *dictator* setting, where each observation is fully assigned to a single member of the household (Section 4). The contributions of this paper include:

(1) for each of the above settings, we formulate an extension of WARP, and

(2) we establish the complexity of testing that extension.

It turns out that testing whether a given dataset S admits a collective rationalization for two members is easy in the general setting, as well as for the dictator setting. It is NP-complete for the private setting. Interestingly, these results are in contrast with known results for SARP and GARP, where all two-member generalizations are hard.

2. The general setting

We present an extension of WARP to test the rationality of two-member collective households under convex preferences, analogue to CARP. In order to describe the rules that define the extension of WARP to collective households with two members, we use the notion of *hypothetical preference* introduced by Cherchye et al. [3, 5]. For member ℓ ($\ell = 1, 2$), we denote by H_0^ℓ the hypothetical preference of that member, and the expression "$q_s \, H_0^\ell \, q_t$" means that we hypothesize that member ℓ *directly prefers* the bundle q_s over the bundle q_t, for $s, t \in N$. Given this notion of hypothetical preference, the extension of WARP to 2-member households is defined as follows.

Definition 6 (2-WARP) *Given is a data set $S := \{(p_t, q_t) : \ t \in N\}$ of a two-member household. A collective rationalization of S exists, i.e., the dataset S satisfies* 2-WARP*, if there exist hypothetical preferences H_0^1, H_0^2 that satisfy the following rules:*

Rule 1: *For each pair of distinct observations $s, t \in N$:*
if $p_s q_s \geq p_s q_t$, then $q_s \, H_0^1 \, q_t$ or $q_s \, H_0^2 \, q_t$.

Rule 2: *For each pair of distinct observations $s, t \in N$:*
if $p_s q_s \geq p_s q_t$ and $q_t \, H_0^\ell \, q_s$ with $\ell \in \{1, 2\}$, then $\neg \left(q_s \, H_0^\ell \, q_t\right)$.

Rule 3: *For each three distinct observations $s, t, u \in N$:*
if $p_s q_s \geq p_s \left(q_t + q_u\right)$ and $q_t \, H_0^r \, q_s$, then $q_s \, H_0^\ell \, q_u$ with $\ell, r \in \{1, 2\}$ and $\ell \neq r$.

Rule 1 states that if bundle q_s was chosen by the household, while bundle q_t was equally attainable (under the prices p_s), then it must be that at least one member prefers bundle q_s over bundle q_t. Rule 2 enforces that the hypothetical preference of each member satisfies WARP. Finally, Rule 3 states that, if the summed bundle $q_t + q_u$ was attainable when q_s was chosen, and one member (say member r) prefers bundle q_t over bundle q_s then it must be the case that the other member prefers q_s over q_u. The problem of testing whether a collective rationalization of S exists is formulated as the following *decision problem*:

Problem 2-WARP
Instance: A data set $S := \{(p_t, q_t) : \ t \in N\}$.
Question: Do there exist hypothetical preferences H_0^1, H_0^2, such that Rules 1-3 hold?

If the data set S contains only three observations, let us say s, t, and u, then the answer to the decision problem is No if and only if the following three inequalities hold: $p_s q_s \geq p_s(q_t + q_u)$, $p_t q_t \geq p_t(q_s + q_u)$, and $p_u q_u \geq p_u(q_s + q_t)$. For data sets containing more than three observations, however, the presence of these three inequalities is not necessary to have a No answer. Indeed, the reader can check that the following inequalities involving four observations, let us say s, t, u, and v, also leads to a No answer to 2-WARP: $p_s q_s \geq p_s q_t$, $p_t q_t \geq p_t(q_s + q_u)$, $p_t q_t \geq p_t(q_s + q_v)$, $p_u q_u \geq p_u(q_t + q_v)$, and $p_v q_v \geq p_v(q_t + q_u)$. Further, we mention that if there is no inequality of the form $p_s q_s \geq p_s(q_t + q_u)$ for all triple s, t, and u in N then we have a YES instance of 2-WARP.

2.1 A graph interpretation of 2-WARP

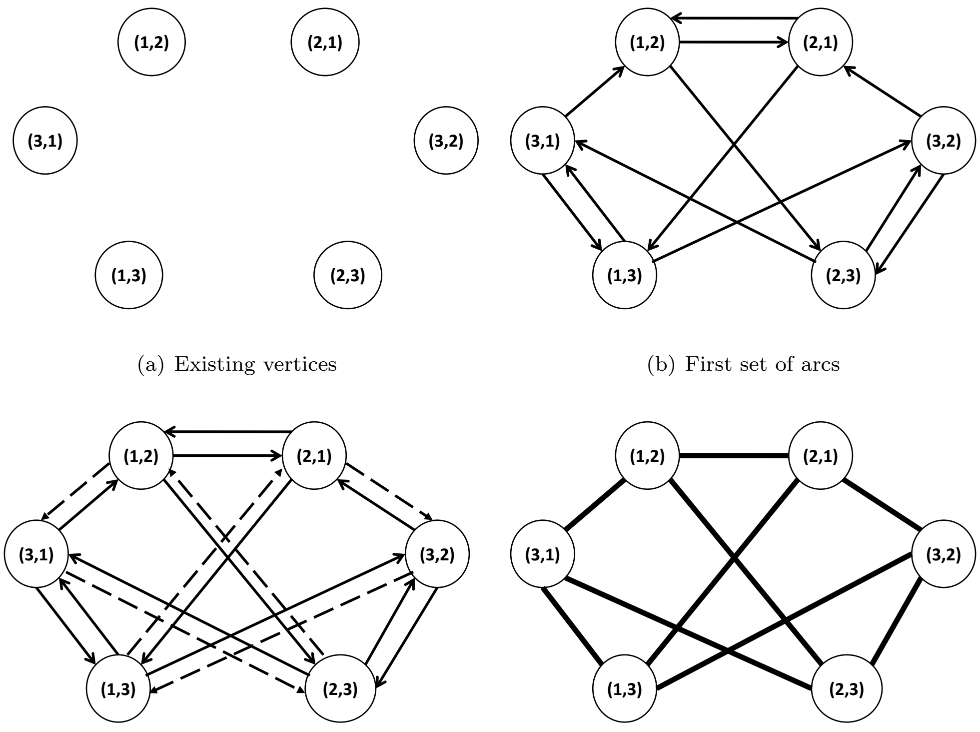
We translate Rules 1–3 into a directed graph setting (see Talla Nobibon et al. [17] for a related contruction). We build a directed graph $G = (V, A)$ from the data set $S := \{(p_t, q_t) : \ t \in N\}$ as follows. A pair of distinct observations (s, t) with $s, t \in N$ represents a vertex in V if and only if both $p_s q_s \geq p_s q_t$ and $p_t q_t \geq p_t q_s$. Notice that V contains $O(n^2)$ vertices and if the vertex (s, t) exists then the vertex (t, s) also exists. The set of arcs A is defined in two steps as follows:

1: First, there is an arc from a vertex (s, t) to a vertex (u, v) whenever $t = u$.

2: Second, for any three distinct observations $s, t, u \in N$ satisfying $p_s q_s \geq p_s(q_t + q_u)$, $p_t q_t \geq p_t q_s$, $p_u q_u \geq p_u q_s$, we have an arc from (s, u) to (t, s), and from (s, t) to (u, s).

Notice that Step 1 ensures that there is an arc from node (s, t) to node (t, s) and vice versa. This graph construction differs from the one used when checking whether a data set of a unitary household satisfies WARP: in that case, a directed graph is built where a vertex corresponds with an observation and there is an arc from s to t if and only if $p_s q_s \geq p_s q_t$. That approach is not considered because it is not quite clear how to deal with inequalities of the form $p_s q_s \geq p_s(q_t + q_u)$.

Given the directed graph $G = (V, A)$ built above, we define the 2-*undirected graph* $G_2 = (V, E)$ associated with G as the undirected graph obtained from G by transforming



(a) Existing vertices

(b) First set of arcs

(c) Final graph G

(d) 2-undirected graph G_2

Figure 1: Illustration of the construction of G and the associated 2-undirected graph G_2

any pair of arcs forming a cycle of length 2 into a single edge (undirected arc); more precisely, $\{v_1, v_2\} \in E$ if and only if $v_1v_2 \in A$ and $v_2v_1 \in A$.

As an illustration of the graph construction, consider a data set with three observations satisfying: $p_1q_1 \geq p_1(q_2 + q_3)$, $p_2q_2 \geq p_2(q_1 + q_3)$, $p_3q_3 \geq p_3(q_1 + q_2)$, $p_2q_2 \geq p_2q_1$, and $p_3q_3 \geq p_3q_1$. This implies the existence of the vertices depicted in Figure 1(a). The arcs stemming from Step 1 appear in Figure 1(b), and the final graph is depicted in Figure 1(c), where the dashed arcs are derived from Step 2. Finally, the 2-undirected graph G_2 associated with G is depicted in Figure 1(d). We have the following result.

Theorem 1 *S is a* Yes *instance of* 2-WARP *if and only if the* 2*-undirected graph G_2 associated with G is bipartite.*

Proof: $\Leftarrow$) Suppose that G_2 is bipartite. Thus, the set of vertices V can be partitioned into two subsets V_1 and V_2 such that each subset induces an independent set. In other words, $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and there is no edge between two vertices of V_1 and no edge between two vertices of V_2. We build the hypothetical preferences H_0^1 and H_0^2 as follows: for every vertex $(s,t) \in V_1$ (respectively $(s,t) \in V_2$) we have $q_s\, H_0^1\, q_t$ $\big($respectively $q_s\, H_0^2\, q_t\big)$. Furthermore, for two distinct observations s and t such that $p_sq_s \geq p_sq_t$ and $(s,t) \notin V$, we set $q_s\, H_0^1\, q_t$ and $q_s\, H_0^2\, q_t$. This completes the definition of H_0^1 and H_0^2. Notice that there is no distinct pair of observations s, t for which we set $q_sH_0^\ell q_t$ and $q_tH_0^\ell q_s$ for some $\ell \in \{1, 2\}$. We now argue that H_0^1 and H_0^2 satisfy Rules 1–3.

Rule 1: Let $s, t \in N$ be two distinct observations such that $p_s q_s \geq p_s q_t$. On the one hand, if $(s,t) \notin V$ then, by construction, $q_s\, H_0^1\, q_t$ and $q_s\, H_0^2\, q_t$. On the other hand, if $(s,t) \in V = V_1 \cup V_2$ then $(s,t) \in V_1$ or $(s,t) \in V_2$, and hence $q_s\, H_0^1\, q_t$ or $q_s\, H_0^2\, q_t$. Thus Rule 1 is satisfied.

Rule 2: As described above, there is no distinct pair of observations $s, t \in N$ for which we set $q_s H_0^\ell q_t$ and $q_t H_0^\ell q_s$ for some $\ell \in \{1,2\}$. Thus Rule 2 is satisfied.

Rule 3: Let $s, t, u \in N$ be three distinct observations such that $p_s q_s \geq p_s(q_t + q_u)$ and $q_t\, H_0^1\, q_s$. There are two cases: (1) if $p_u q_u < p_u q_s$ then $(s,u) \notin V$ and, since $p_s q_s \geq p_s q_u$, we have by construction of H_0^ℓ, $q_s\, H_0^1\, q_u$ and $q_s\, H_0^2\, q_u$, and we are done; (2) if $p_u q_u \geq p_u q_s$ then $(s,u) \in V$. Let us now argue by contradiction that $(t,s) \in V$. Indeed, if $(t,s) \notin V$, then $(s,t) \notin V$. That however, is impossible since $p_s q_s \geq p_s q_t$, and we would have had by construction $q_s\, H_0^1\, q_t$ and $q_s\, H_0^2\, q_t$, which cannot be reconciled with $q_t\, H_0^1\, q_s$. Thus $(t,s) \in V$, and in fact, since $q_t\, H_0^1\, q_s$, $(t,s) \in V_1$. Following the construction of G, we have an arc from (t,s) to (s,u) and an arc from (s,u) to (t,s) (because $p_s q_s \geq p_s(q_t + q_u), p_t q_t \geq p_t q_s, p_u q_u \geq p_u q_s$). Therefore, there is an edge between the vertices (t,s) and (s,u) in G_2, and we conclude that $(s,u) \in V_2$, which implies that $q_s\, H_0^2\, q_u$. This completes the verification of Rule 3.

$\Rightarrow$) Now, we suppose that S is a Yes instance of 2-WARP; there exist H_0^1 and H_0^2 satisfying Rules 1–3. We want to show that the 2-undirected graph G_2 is bipartite. In other words, we want to partition V into two subsets V_1 and V_2 such that there is no edge between two vertices of V_1 and no edge between two vertices of V_2.

Given H_0^1 and H_0^2 we set the vertices in V_1 (respectively in V_2) as follows: a vertex $(s,t) \in V$ belongs to V_1 (respectively to V_2) if $q_s\, H_0^1\, q_t$ $\big($respectively $q_s\, H_0^2\, q_t\big)$. It is not difficult to see that $V_1 \cap V_2 = \emptyset$ and that any vertex in V is either in V_1 or in V_2. Hence, V_1 and V_2 constitute a valid partition of V. We argue, by contradiction, that V_1 and V_2 induce independent sets. Without loss of generality, suppose V_1 is not an independent set. There exist two vertices (s,t) and (u,v) in V_1 with an edge between them in G_2. Thus, in the graph G there is an arc from (s,t) to (u,v), and from (u,v) to (s,t). If both arcs originate from Step 1, we have $u = t$ and $v = s$, which implies $(s,t) \in V_1$, and $(t,s) \in V_1$ which can only happen if $q_s\, H_0^1\, q_t$ and $q_t\, H_0^1\, q_s$; this, however, contradicts Rule 2 for H_0^1. If both arcs originate from Step 2, we also have $u = t$ and $v = s$, and the same argument applies. Hence, one arc originates from Step 1 and one arc originates from Step 2. Without loss of generality, we can assume that the arc from (s,t) to (u,v) comes from Step 1, while the arc from (u,v) to (s,t) comes from Step 2. This implies that $u = t$, and apparently $p_t q_t \geq p_t(q_s + q_v)$. Since $q_s\, H_0^1\, q_t$, Rule 3 implies that $q_t\, H_0^2\, q_v$. By hypothesis, we have $q_t\, H_0^1\, q_v$ and $p_v q_v \geq p_v q_t$ (because $(t,v) \in V_1$). From Rule 1 we know that $q_v\, H_0^1\, q_t$ or $q_v\, H_0^2\, q_t$. This, together with $q_t\, H_0^1\, q_v$ and $q_t\, H_0^2\, q_v$, implies that either $q_t\, H_0^1\, q_v$ and $q_v\, H_0^1\, q_t$ or $q_t\, H_0^2\, q_v$ and $q_v\, H_0^2\, q_t$. In the first case, H_0^1 violates Rule 2 whereas in the second case H_0^2 violates Rule 2. In both cases, we have a contradiction with Rule 2. This concludes the proof of Theorem 1.

2.2 Algorithm for 2-WARP

We present an algorithm for 2-WARP that is based on Theorem 1. The pseudocode is described by Algorithm 1.

It is clear that each of the three steps of Algorithm 1 can be done in polynomial time. Thus, we have the following result:

Algorithm 1 *Algorithm for* 2-WARP

1: build the directed graph G from the data set S
2: build the 2-undirected graph G_2 associated with G
3: **if** G_2 is bipartite **then** return YES, **else** return NO

Theorem 2 *Algorithm 1 solves* 2-WARP *in polynomial time.*

3. The private setting

We consider a household with 2 members acting in an economy with m goods that can only be consumed privately without externalities. The extension to a household with $k \geq 2$ members is immediate. For each observation $t \in N$, a *feasible personalized quantity vector* is a pair (q_t^1, q_t^2); this pair can be seen as one of the infinitely many feasible split ups of the observed quantity vector q_t. More concrete, $q_t = q_t^1 + q_t^2$ for each $t \in N$. The true split up of q_t is unobserved. For each member ℓ ($\ell = 1, 2$) we define the *personalized consumption data set* by $S_\ell = \{(p_t, q_t^\ell) : t \in N\}$.

Definition 7 *(private* 2-WARP*) Given is a data set $S = \{(p_t, q_t) : t \in N\}$ of a two-member household. We say that S is consistent with the* 2-member egoistic collective consumption model*, i.e., S satisfies* private 2-WARP*, if and only if:*

Condition 1: *For each $t \in N$ there exist q_t^1, $q_t^2 \in \mathbb{R}_+^m$ such that $q_t = q_t^1 + q_t^2$, and*

Condition 2: *For each member $\ell \in \{1, 2\}$, the set $S_\ell = \{(p_t, q_t^\ell) : t \in N\}$ satisfies* WARP.

This problem can be phrased as the following decision problem:

Problem: private 2-WARP
Instance: A data set $S = \{(p_t, q_t) : t \in N\}$.
Question: Do there exist q_t^1, $q_t^2 \in \mathbb{R}_+^m$ satisfying $q_t = q_t^1 + q_t^2$ for each $t \in N$ such that for $\ell = 1, 2$, the set $S_\ell = \{(p_t, q_t^\ell) : t \in N\}$ satisfies WARP?

It turns out that answering this question is NP-complete even for two members in the household.

Theorem 3 *Testing private* 2-WARP *is NP-complete.*

Proof: We use a reduction from MONOTONE NOT-ALL-EQUAL 3-SAT (which is known to be NP-complete).
Instance: A set of variables $X = \{x_1, x_2, \ldots, x_n\}$ and a set of clauses $C = \{c_1, c_2, \ldots, c_m\}$ with each clause consisting of 3 positive literals.
Question: Does there exist a truth-assignment so that for each clause, either one or two of the literals are TRUE?

It is not difficult to see that private 2-WARP belongs to the class NP. The rest of this proof is structured as follows: given an arbitrary instance of MNAE 3-SAT, we first build an instance of private 2-WARP and next, we prove that we have a YES instance of MNAE 3-SAT if and only if the constructed instance of private 2-WARP is a YES instance.

Consider an arbitrary instance $X = \{x_1, x_2, \ldots, x_n\}$ and $C = \{c_1, c_2, \ldots, c_m\}$ of MNAE 3-SAT. We build an instance of private 2-WARP using $3n + 4$ goods and $2n + 2m + 3$ observations. We next describe the quantity and the price of goods for each observation.

We use $\epsilon = \frac{1}{4n}$ and $M = n + 1$. The first block of $2n$ observations corresponds to the variables and is given by:

$$
\begin{array}{llr}
q_1 & = (0,0,|0,0,|0,\dots,0,|1,0,\dots,0,|0,\dots,0)\,; & p_1 = (M,M,|M,M,|1,\epsilon,\dots,\epsilon,|1,M,\dots,M,|\epsilon,M,\dots,M) \\
q_2 & = (0,0,|0,0,|0,\dots,0,|0,1,\dots,0,|0,\dots,0)\,; & p_2 = (M,M,|M,M,|\epsilon,1,\dots,\epsilon,|M,1,\dots,M,|M,\epsilon,\dots,M) \\
\vdots & & \\
q_n & = (0,0,|0,0,|0,\dots,0,|0,0,\dots,1,|0,\dots,0)\,; & p_n = (M,M,|M,M,|\epsilon,\epsilon,\dots,1,|M,M,\dots,1,|M,M,\dots,\epsilon) \\
q_{n+1} & = (0,0,|0,0,|0,\dots,0,|0,\dots,0,|1,0,\dots,0)\,; & p_{n+1} = (M,M,|M,M,|M,\dots,M,|\epsilon,M,\dots,M,|1,M,\dots,M) \\
\vdots & & \\
q_{2n} & = (0,0,|0,0,|0,\dots,0,|0,\dots,0,|0,\dots,0,1)\,; & p_{2n} = (M,M,|M,M,|M,\dots,M,|M,M,\dots,\epsilon,|M,M,\dots,1)
\end{array}
$$

The second block of $2m$ observations corresponds to the clauses. For each clause $c_a = \{x_i, x_j, x_k\}$, we have the observations $2n + a$ and $2n + m + a$ $(a = 1, \dots, m)$.

$$
\begin{array}{llr}
q_{2n+a} & = (1,0,|0,0,|0,\dots,0,|0,\dots,0,|0,\dots,0)\,; & p_{2n+a} = (1,\epsilon,|M,M,|\,\{\epsilon,1\}\,|M,\dots,M,|M,\dots,M) \\
q_{2n+m+a} & = (0,1,|0,0,|0,\dots,0,|0,\dots,0,|0,\dots,0)\,; & p_{2n+m+a} = (\epsilon,1,|M,M,|\,\{\epsilon,1\}\,|M,\dots,M,|M,\dots,M)
\end{array}
$$

The prices of the goods corresponding to variables x_i, x_j and x_k equal 1, and the prices of the goods corresponding to other variables equal ϵ. Finally, we have observations $2n + 2m + 1$, $2n + 2m + 2$, $2n + 2m + 3$.

$$
\begin{array}{llr}
q_{2n+2m+1} & = (0,0,|1,0,|0,\dots,0,|0,\dots,0,|0,\dots,0)\,; & p_{2n+2m+1} = (M,M,|1,\epsilon,|\epsilon,\dots,\epsilon,|M,\dots,M,|M,\dots,M) \\
q_{2n+2m+2} & = (0,0,|0,1,|0,\dots,0,|0,\dots,0,|0,\dots,0)\,; & p_{2n+2m+2} = (M,M,|\epsilon,1,|\epsilon,\dots,\epsilon,|M,\dots,M,|M,\dots,M) \\
q_{2n+2m+3} & = (0,0,|0,0,|1,\dots,1,|0,\dots,0,|0,\dots,0)\,; & p_{2n+2m+3} = (\epsilon,\epsilon,|n-1,n-1,|1,\dots,1,|\epsilon,\dots,\epsilon,|\epsilon,\dots,\epsilon)
\end{array}
$$

Before embarking on the proof, let us describe the main idea. Consider the n goods, $5, 6, 7, \dots, n + 4$ in observation $2n + 2m + 3$. Each of these goods corresponds to a variable in the instance of MNAE-3SAT. We will argue that each of these n goods is allocated for a large part (i.e. $\geq \frac{3}{4}$) to some member $\ell \in \{1, 2\}$. This is akin to setting the corresponding variable to TRUE (if the good goes for the larger part to member 1), or to FALSE (if the good goes for the larger part to member 2). Of course it remains to show that this is a satisfying truth assignment.
According to Definition 1, recall that we say that member $\ell \in \{1, 2\}$ directly prefers observation a over observation b, when we have $p_a q_a^\ell \geq p_a q_b^\ell$ with $a, b \in S$.

Claim 1 *If $p_a q_a \geq p_a q_b$ for some $a, b \in S, a \neq b$, then there exists a member $\ell \in \{1, 2\}$, who directly prefers observation a over observation b.*

Proof: Consider any split of q_a into q_a^1, q_a^2, and q_b into q_b^1, q_b^2, i.e., let $q_a^1 + q_a^2 = q_a$ and $q_b^1 + q_b^2 = q_b$. Since $p_a q_a \geq p_a q_b$, it follows that $p_a(q_a^1 + q_a^2) \geq p_a(q_b^1 + q_b^2)$. Hence, either $p_a q_a^1 \geq p_a q_b^1$ or $p_a q_a^2 \geq p_a q_b^2$ (or both). □

Notice that, apart from bundle $q_{2n+2m+3}$, all other bundles are unit vectors. We will use $q_{i,j}(p_{i,j})$ to denote the quantity (price) of good j in observation i, $i = 1, \dots, 2n+2m+3$, $j = 1, \dots, 3n + 4$. We now exhibit a trick that we will use throughout the proof. Consider a hypothetical dataset, containing the observations a and b as follows:

$$
\begin{array}{ll}
q_a = (1, 0), & p_a = (1, \epsilon) \\
q_b = (0, 1), & p_b = (\epsilon, 1)
\end{array}
$$

We say that the split of a bundle is *extreme* if each unit good of that bundle goes to one (of the two) members with fraction at least $1 - \epsilon$.

Claim 2 *In any feasible solution of some data set containing observations a and b, the split of bundles q_a and q_b is extreme.*

Proof: Clearly, we have both $p_a q_a > p_a q_b$ and $p_b q_b > p_b q_a$. So using claim 1, it follows that one member directly prefers observation a over b, and one member directly prefers b over a. Thus in any feasible solution these members must be different (otherwise private 2-WARP is violated). Let us assume, without loss of generality, that member 1 directly prefers a over b and does not directly prefer b over a. Let α be the fraction of bundle a allocated to member 1, and β the fraction of bundle b allocated to this member. Since member 1 does not directly prefer b over a, we find:

$$p_b q_b^1 < p_b q_a^1 \Rightarrow \beta < \epsilon\alpha$$

Since $\alpha \leq 1$, we conclude $\beta < \epsilon = \frac{1}{4n}$. Likewise, since member 2 does not directly prefer a over b, we find:

$$p_a q_a^2 < p_a q_b^2 \Rightarrow 1 - \alpha < \epsilon(1-\beta) \Rightarrow \alpha > 1 - \epsilon = \frac{4n-1}{4n}.$$

Claim 2 follows. □

Clearly, claim 2 is applicable to any pair of observations involving bundles that are unit vectors, and price vectors that feature price ϵ and price 1.

Let us now establish the validity of the following two inequalities.

Claim 3 *In any feasible solution to this instance of private* 2-WARP*, we have for $\ell = 1, 2$: $p_{2n+2m+3} q_{2n+2m+3}^{\ell} > 1$.*

Proof: Observe that claim 2 is applicable to observations $2n+2m+1$ and $2n+2m+3$. Thus the split of the bundles $q_{2n+2m+1}$ and $q_{2n+2m+2}$ is extreme. Let us assume, without loss of generality, that good 2 is allocated to member 1 with fraction at least $1-\epsilon$, while good 4 is allocated to member 2 with fraction $1-\epsilon$. Thus:

$$p_{2n+2m+1} q_{2n+2m+1}^1 \geq 1 - \epsilon = \frac{4n-1}{4n} \geq \frac{1}{4} = n\frac{1}{4n} = n\epsilon \geq p_{2n+2m+3} q_{2n+2m+3}. \quad (1)$$

It follows that member 1 prefers, in any feasible solution, observation $2n+2m+1$ over $2n+2m+3$. Then, in order to satisfy private 2-WARP, we must have:

$$p_{2n+2m+3} q_{2n+2m+3}^1 < p_{2n+2m+3} q_{2n+2m+1}^1 \leq p_{2n+2m+3} q_{2n+2m+1} \Rightarrow \quad (2)$$

$$\sum_{i=1}^{n} q_{2n+2m+3,4+i}^1 < n - 1. \quad (3)$$

Since, for $a \in S$, $q_a^2 = q_a - q_a^1$, we derive, using (3):

$$\sum_{i=1}^{n} q_{2n+2m+3,4+i}^2 = n - \sum_{i=1}^{n} q_{2n+2m+3,4+i}^1 > 1. \quad (4)$$

Finally, since $p_{2n+2m+3,i} = 1$ for $i = 5, 6, \ldots, n+4$, it follows that (4) can be written as:

$$p_{2n+2m+3} q_{2n+2m+3}^2 > 1.$$

A similar reasoning involving member 2 and observations $2n+2m+2$ and $2n+2m+3$ leads to:

$$p_{2n+2m+3} q_{2n+2m+3}^1 > 1.$$

□

To proceed, let us consider observation i, $(1 \leq i \leq n)$, and observations $2n + 2m + 3$. Using claim 3, we observe:

$$p_{2n+2m+3}q^{\ell}_{2n+2m+3} > 1 > p_{2n+2m+3}q_i \qquad \text{for } \ell = 1, 2, 1 \leq i \leq n. \tag{5}$$

Thus, no matter the split of q_i into q^1_i and q^2_i, both member 1 and member 2 each directly prefer observation $2n + 2m + 3$ over observation $i = 1, \ldots, n$. Since we have a YES-instance of 2-WARP, we know that then, for $\ell = 1, 2$:

$$p_i q^{\ell}_i \leq p_i q^{\ell}_{2n+2m+3}. \tag{6}$$

Observe that claim 2 is applicable to observations i and $n + i$. Thus, the split of q_i and q_{n+i} is extreme. Hence, there is a member ℓ for which:

$$p_i q^{\ell}_i \geq 1 - \epsilon. \tag{7}$$

Inequalities(6) and (7) imply that the split of $q_{2n+2m+3}$ is such that:

$$p_i q^{\ell}_{2n+2m+3} \geq 1 - \epsilon. \tag{8}$$

Consider the vectors p_i and $q_{2n+2m+3}$, $1 \leq i \leq n$. It follows that:

$$p_i q^{\ell}_{2n+2m+3} = \epsilon \sum_{j=5, j \neq 4+i}^{n+4} q^{\ell}_{2n+2m+3,j} + q^{\ell}_{2n+2m+3,4+i}. \tag{9}$$

Also:

$$\sum_{j=5, j \neq 4+i}^{n+4} q^{\ell}_{2n+2m+3,j} \leq \sum_{j=5, j \neq 4+i}^{n+4} q_{2n+2m+3,j} = n - 1. \tag{10}$$

Rewriting (9), and using inequalities (8) and (10) gives for each $i = 1, \ldots, n$:

$$q^{\ell}_{2n+2m+3,4+i} = p_i q^{\ell}_{2n+2m+3} - \epsilon \sum_{j=5, j \neq 4+i}^{n+4} q^{\ell}_{2n+2m+3,j} \geq 1 - \epsilon - \epsilon(n-1) = 1 - \frac{n}{4n} = \frac{3}{4}. \tag{11}$$

Concluding, each good $i = 5, 6, \ldots, n + 4$ in observation $2n + 2m + 3$ is allocated for over $\frac{3}{4}$ to some member $\ell \in \{1, 2\}$.

Finally, we look at the two observations corresponding to each clause $j = 1, \ldots, m$. It is clear that each members directly prefers observation $2n + 2m + 3$ over both observation $2n + j$ and $2n + m + j$. Observe also that claim 2 is applicable to observations $2n + j$ and $2n + 2m + j$. Thus, in order not to have a violation of private 2-WARP, member ℓ should not prefer $2n + j$ over $2n + 2m + 3$. Thus, for each $\ell = 1, 2$:

$$p_{2n+j} q^{\ell}_{2n+j} < p_{2n+j} q^{\ell}_{2n+2m+3}. \tag{12}$$

Since (without loss of generality), for member 1, we have $p_{2n+j}q^1_{2n+j} \geq 1-\epsilon$, and thus we have using (12):

$$p_{2n+j}q^1_{2n+2m+3} > 1-\epsilon. \tag{13}$$

This means that one of the three goods associated to clause j is allocated over $\frac{3}{4}$ to member 1. We argue by contradiction. Indeed, in case none of the three goods of clause j are allocated over $\frac{3}{4}$ to member 1, then they are allocated for at most $\frac{1}{4}$ to member 1. Then,

$$p_{2n+j}q^1_{2n+2m+3} \leq 3\frac{1}{4} + (n-3)\epsilon = \frac{3}{4} + \frac{n-3}{4n} = \frac{4n-3}{4n} < \frac{4n-1}{4n}. \tag{14}$$

Thus we would have $p_{2n+j}q^1_{2n+2m+3} < 1-\epsilon$, contradicting (13). Therefore, at least one of the goods associated with j is allocated over $\frac{3}{4}$ to member 1. Clearly, a similar reasoning involving $2n+2m+j$ and member 2 implies that one of these three goods must be allocated over $\frac{3}{4}$ to member 2.

In conclusion, we now know the following about any valid allocation of observation $2n+2m+3$ which satisfies private 2-WARP. First, that each good is split up in a large and a small allocation for the different members. Secondly, that for each clause and each member, there is at least one of the goods associated with the variables that has a large allocation. A valid truth assignment for MNAE 3-SAT can now be found as follows. If a good is largely allocated to member 1, the variable is set to TRUE, if a good is allocated to member 2, the variable is FALSE.

If we have a Yes-instance of MNAE 3-SAT, an allocation of goods which satisfies private 2-WARP exists. For observation $2n+2m+3$, fully assign each good associated with a TRUE variable to member 1, and each good associated with a FALSE variable to member 2. Likewise, fully assign the bundle i to member 1 if x_i is TRUE and to member 2 if it is FALSE. Furthermore, for all $j = 1, \ldots, m$, fully assign bundles $2n+j$ to member 1 and all $2n+m+j$ to member 2. Finally, fully assign $2n+2m+1$ to member 1 and $2n+2m+2$ to member 2. It can be easily checked that such an allocation satisfies private 2-WARP. □

4. The dictator setting

In this setting, each member is considered as a *dictator* in the sense that when he or she has decided to consume a given bundle (observation), he or she will consume the entire bundle alone. Notice that in the unitary setting, the decision maker can be seen as the only dictator of the household. The situation-dependent dictatorship can be interpreted as a direct collective generalization of the unitary decision model. More precisely, the new model considers k separate decision makers in the household ($k \geq 2$); each is (fully) responsible for a subset of observations and the subsets associated with two different decision makers are pairwise disjoint. As a consequence, the sufficiency condition for rationalizing the observed consumption data set is the existence of a partition of that set into k subsets such that each subset is consistent with the considered unitary model (in our case WARP). In other words, each individual dictator must act consistent with the unitary rationality condition for those quantities for which she or he is (fully) responsible.

Formally, we consider a household with k members acting in an economy with m goods and the data set $S = \{(p_t, q_t) : t \in N\}$.

Definition 8 *(dictator k-WARP) Given is a data set $S = \{(p_t, q_t) : t \in N\}$ and an integer k. We say that S is* consistent with k dictators *if and only if N can be partitioned into k*

pairwise disjoint subsets N_ℓ such that each subset $S_\ell := \{(p_t, q_t) : \ t \in N_\ell\}$ ($\ell = 1, \ldots, k$) satisfies WARP.

This problem can be phrased as the following decision problem:

Problem: dictator k-WARP
Instance: A data set $S = \{(p_t, q_t) : \ t \in N\}$, and an integer k.
Question: Can N can be partitioned into k pairwise disjoint subsets N_ℓ such that each subset $S_\ell := \{(p_t, q_t) : \ t \in N_\ell\}$ ($\ell = 1, \ldots, k$) satisfies WARP?

It turns out that answering this question is NP-complete for three members in the household, and is solvable in polynomial time for two members in the household. In order to prove this, we introduce the undirected graph $H = (V, E)$, which is constructed as follows. Each vertex corresponds with an observation in S and there is a edge between vertex s and vertex t if and only if $p_s q_s \geq p_s q_t$ and $p_t q_t \geq p_t q_s$.

Theorem 4 *The graph H can be colored using k colors if and only if S is consistent with k dictators.*

Proof: Given a dataset S, consistent with k dictators, there exist disjoint subsets $S_1, S_2, \ldots, S_k$, such that for any two observations $a, b \in S_i$ either $p_s q_s < p_s q_t$ and/or $p_t q_t < p_t q_s$. The graph can thus be k colored by giving the same color to all vertices associated with observations in the same subset. Furthermore, given an arbitrary graph used as instance for graph coloring, it is not difficult to see that, by using a good for each vertex in the graph, we can choose prices and quantities such that an instance of dictator k-WARP arises which has $p_s q_s \geq p_s q_t$ and $p_s q_s \geq p_s q_t$ whenever nodes s and t are connected. Such a construction is described in Deb and Pai [10]. Next, it should be clear that a feasible k-coloring corresponds to a partition of the dataset consistent with k dictators. □

Theorem 4 immediately implies that any algorithm for graph coloring problems can be used for testing whether a given data set S is consistent with k dictators. The following two theorems thus follow immediately.

Theorem 5 *Testing dictator* 2-WARP *can be done in polynomial time.*

And

Theorem 6 *Testing dictator k-*WARP *is NP-complete for each fixed $k \geq 3$.*

5. Conclusion

We studied three different possible extensions of the weak axiom of revealed preference (WARP) to households that consist of two members. We proved that for the general setting setting, testing 2-WARP can be done in polynomial time. This is also true for the dictator setting; however, for the private setting, the corresponding problem is proven to be NP-complete.

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Frits C.R. Spieksma
Faculty of Economics and Business - ORSTAT
KU Leuven
Leuven, Belgium
Email: `frits.spieksma@kuleuven.be`

Bart Smeulders
Faculty of Economics and Business - ORSTAT
KU Leuven
Leuven, Belgium
Email: `bart.smeulders@kuleuven.be`

Laurens Cherchye
Faculty of Economics and Business - CES
KU Leuven
Leuven, Belgium
Email: `laurens.cherchye@kuleuven.be`

Bram De Rock
ECARES-ECORE, Université Libre de Bruxelles
Brussels, Belgium
Email: `bderock@ulb.ac.be`

Fabrice Talla Nobibon
FEDEX
Brussels, Belgium
Email: