Two-sided problems with choice functions, matroids and lattices

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Summer School on
Matching Problems, Markets, and Mechanisms
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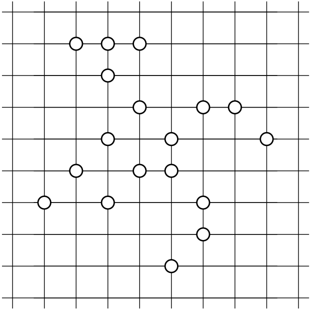
A competition problem

Prove that any finite subset H of the planar grid has a subset K with the property that

1. any vertical or horizontal line intersects K in at most 2 points,
2. any point of $H \setminus K$ lies on a vertical or horizontal segment determined by K.



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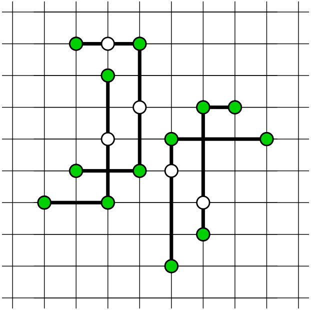

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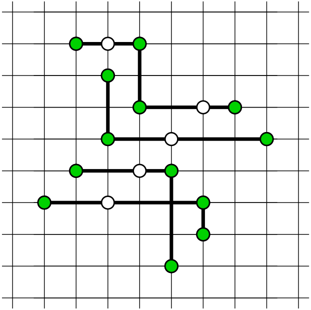
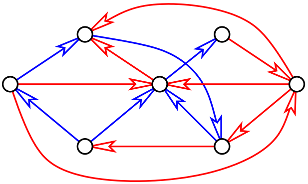


Yet another competition problem

In a certain country intercity traffic is served by trains and coaches. Both the railway and bus company runs its lines between certain pairs of cities, but between two cities there migth be no line that goes both ways. We know that no matter how we pick two cities, one can travel from one city to the other either by bus or by train, perhaps with changes, and the opposite travel is not necessarily possible. Prove that there exists a city from which any other city is reachable with possible changes by using only one mean of transport such that for different cities we might need different kind of transport.



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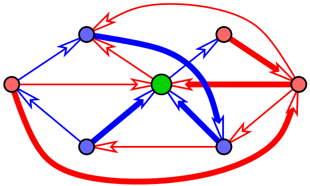

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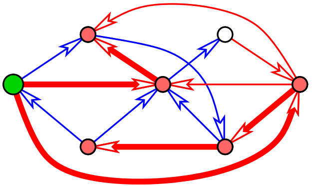

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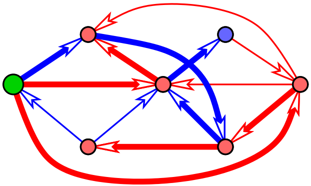

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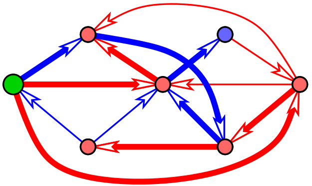

Hey! Who cares about obscure competion problems??? We wanna learn about two-sided markets. Give us value for the money!!!

Two-sided markets: college admissions and graphs

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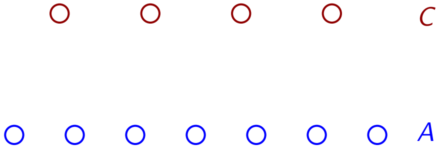

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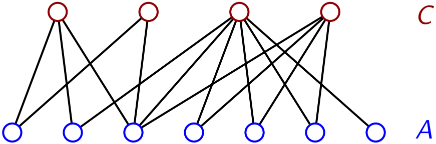

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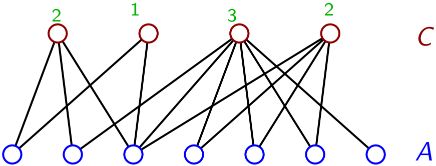

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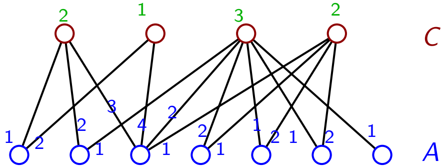

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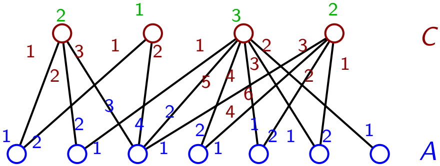

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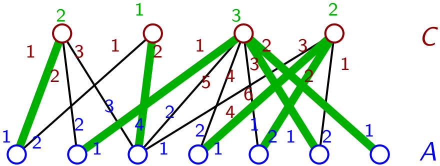

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An **admission scheme** or **assignment** is a set of applications that assigns each applicant to at most 1 college and each college c to at most $q(c)$ applicants.

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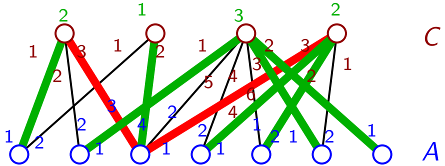

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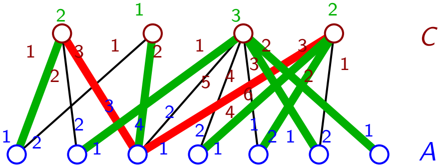

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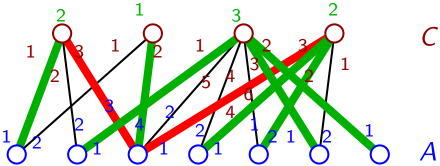

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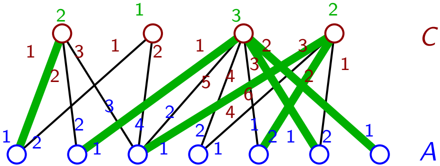

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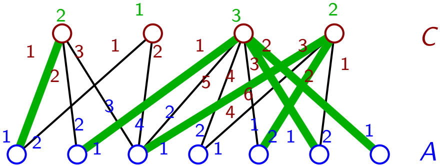

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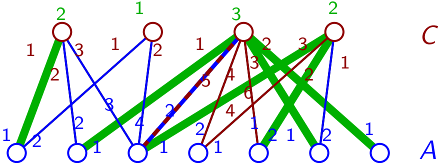

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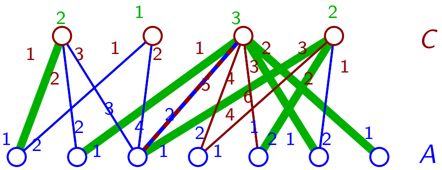

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Property:
If students are offered $S \cup \mathcal{D}_A(S)$ then they choose S ,
if colleges are offered $S \cup \mathcal{D}_C(S)$ then they choose S.
That is, $\mathcal{C}_A(S \cup \mathcal{D}_A(S)) = S$ and $\mathcal{C}_C(S \cup \mathcal{D}_C(S)) = S$.

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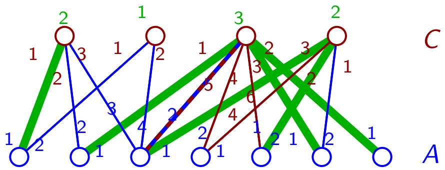

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Goal: A choice-function based approach to two-sided markets.

Stability and choice functions

Contract: application (edge of the underlying graph).

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Choice funcion model: applicants and colleges have choice functions on the contracts: $\mathcal{C}_A(F) \subseteq F$ and $\mathcal{C}_C(F) \subseteq F \quad \forall F \subseteq E$.

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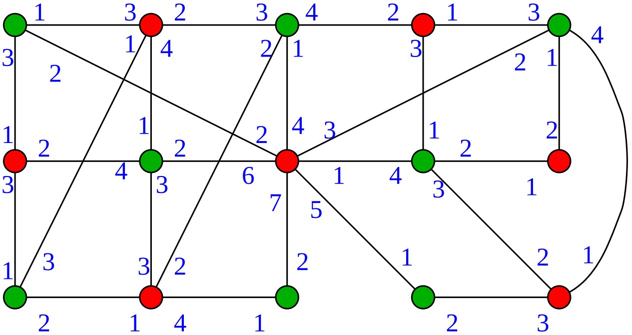
Fact: If $\mathcal{C}$ is substitutable and increasing then $\mathcal{C}$ is PI.

The deferred acceptance algorithm

Gale-Shapley Theorem: There always exists a stable matching.

The deferred acceptance algorithm



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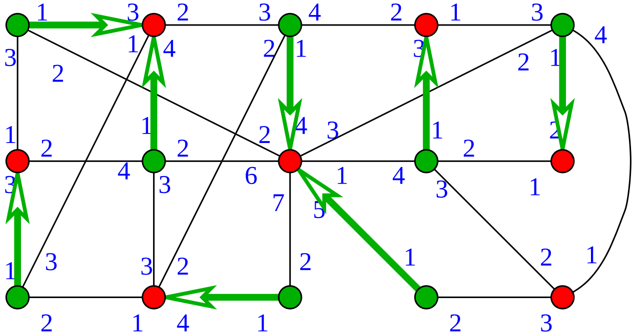

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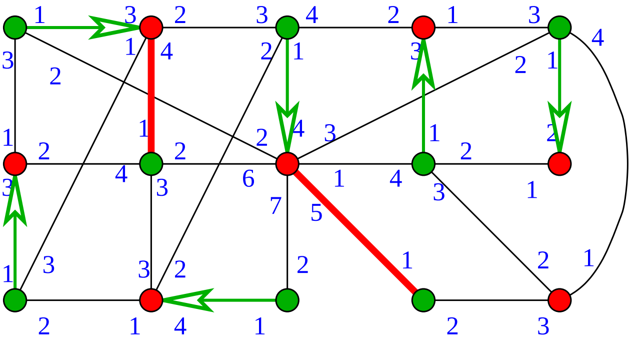

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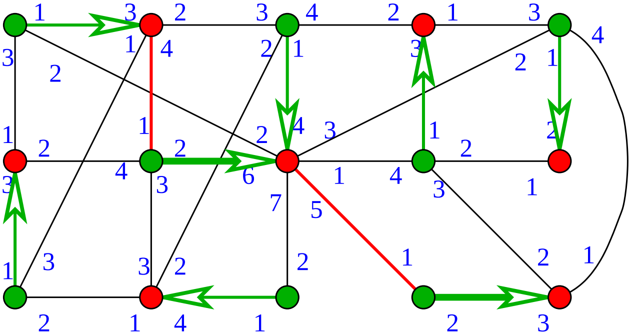

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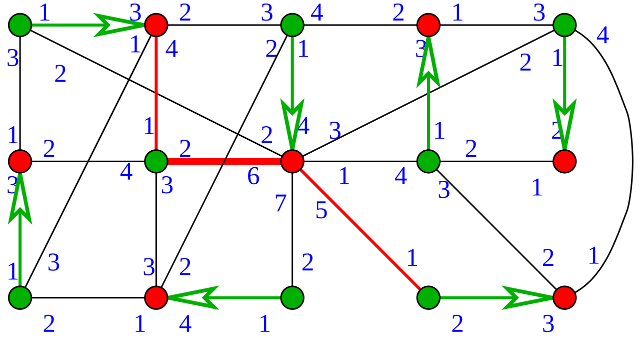

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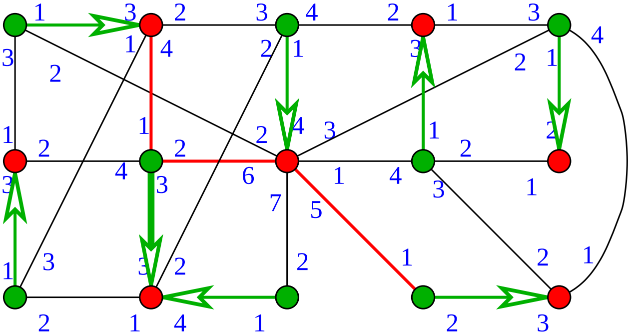

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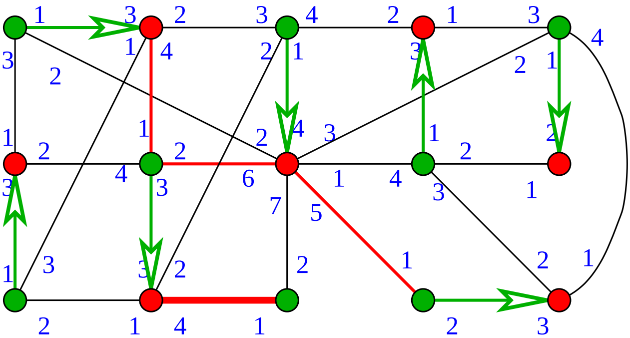

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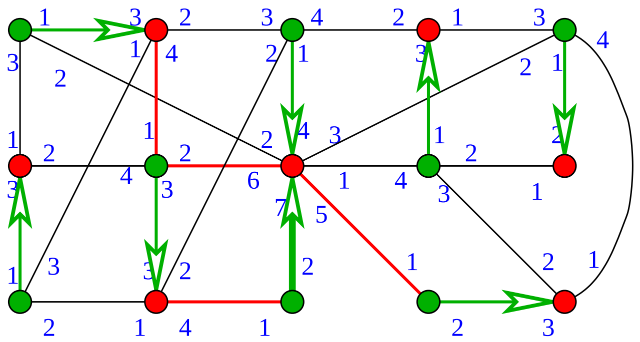

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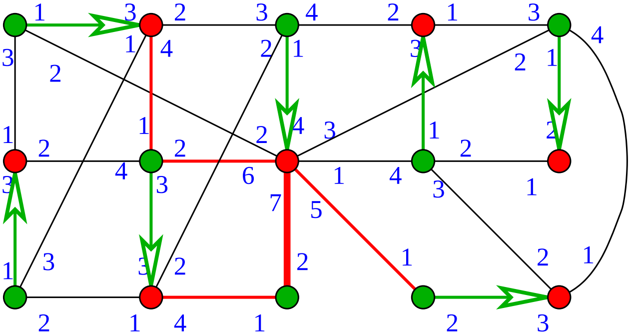

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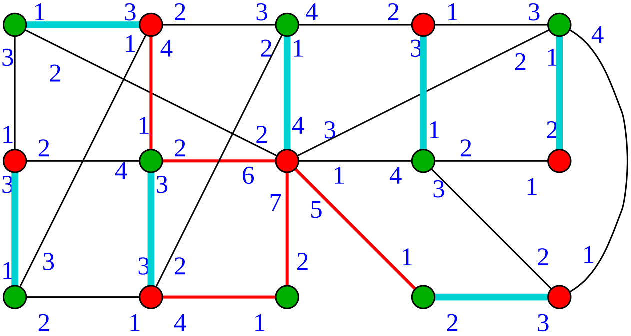

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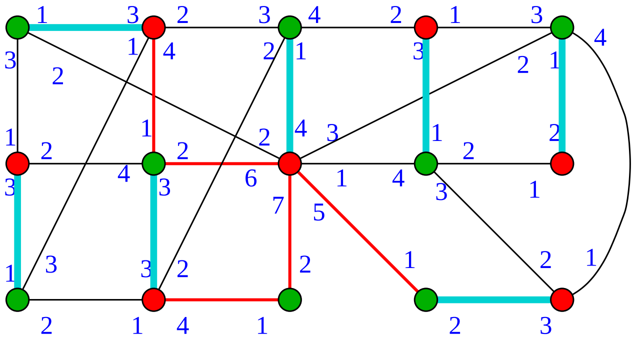

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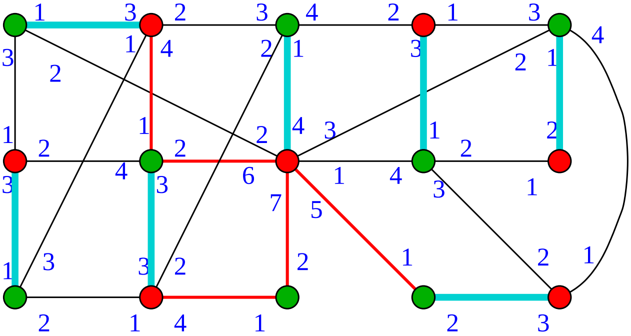

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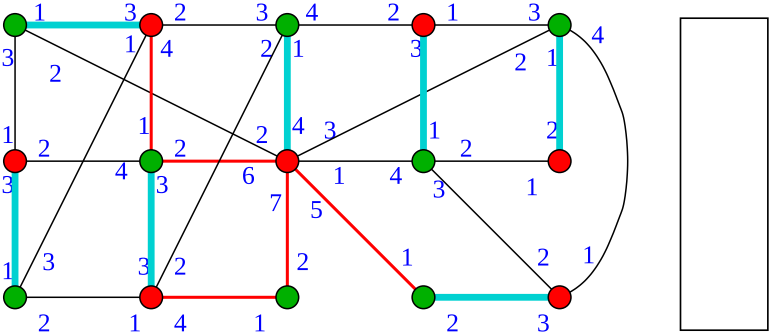

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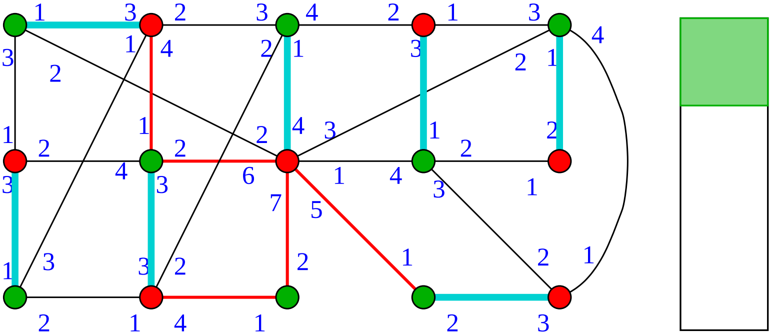

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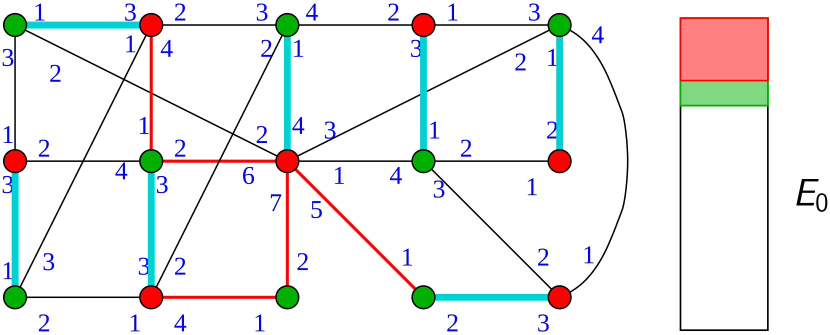

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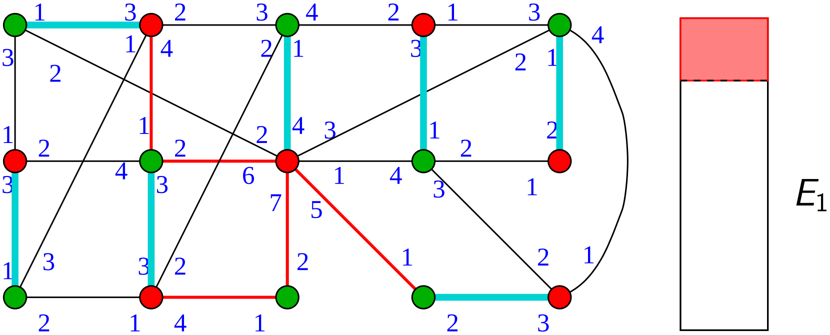

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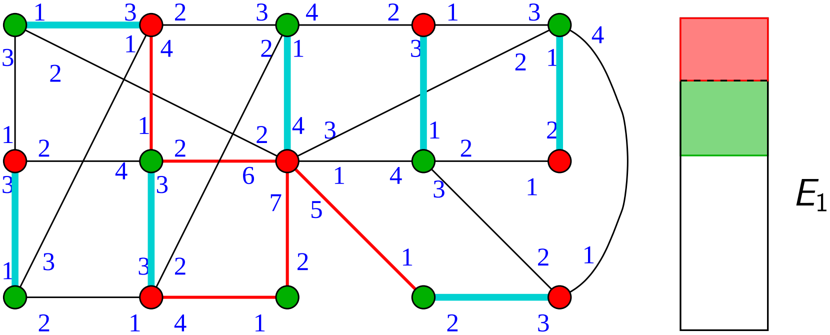

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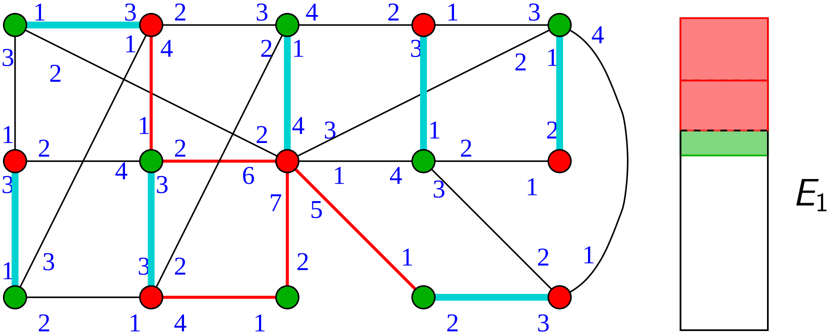

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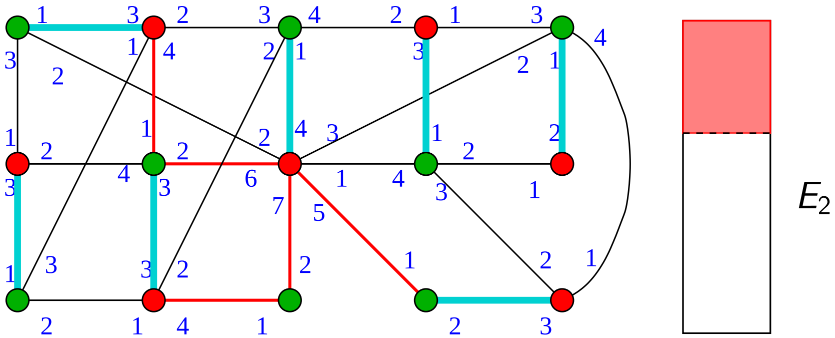

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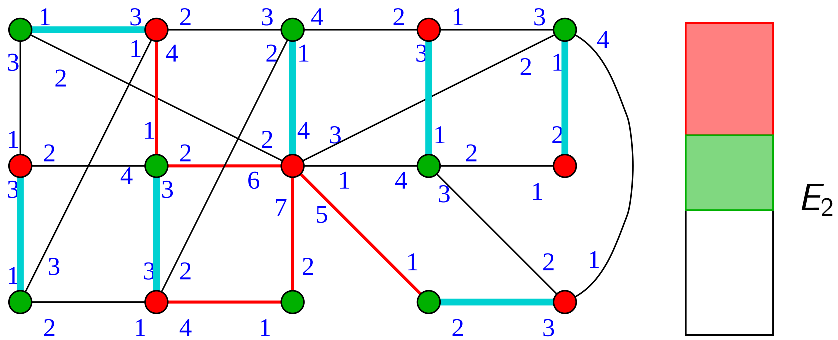

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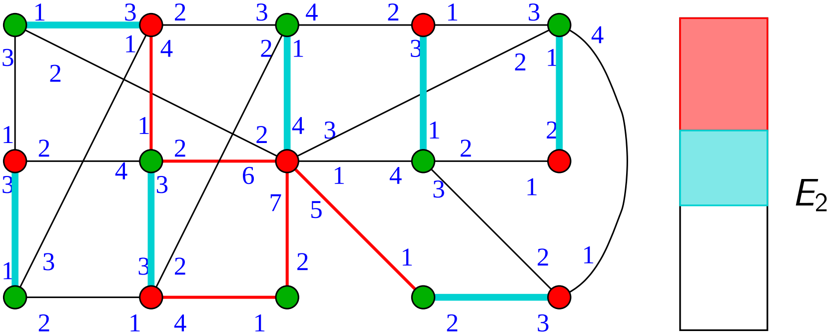

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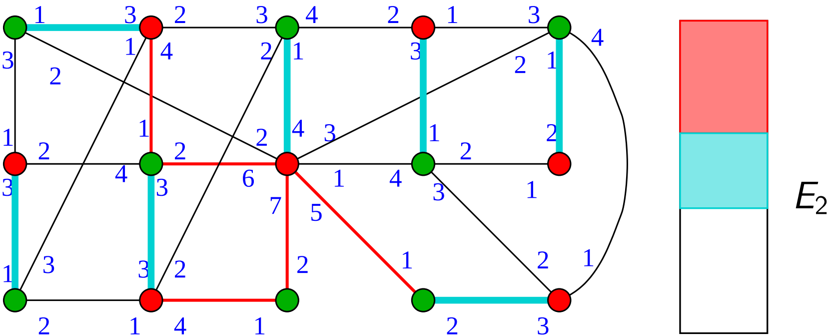

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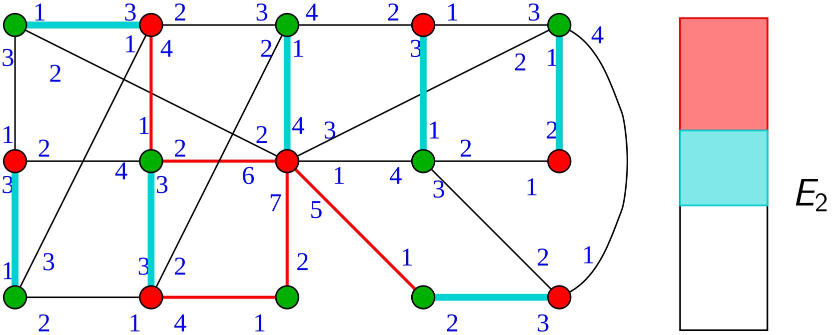

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Stupid question: What makes this algorithm work?

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Observation: The Gale-Shapely algorithm is an iteration of a monotone function. By definition, $E_{i+1} = \mathcal{F}(E_i)$, where $\mathcal{F}(X) = X \setminus (\mathcal{C}_A(X) \setminus \mathcal{C}_C(\mathcal{C}_A(X))$ =(by PI)= $E \setminus \overline{\mathcal{C}}_C(E \setminus \overline{\mathcal{C}}_A(X))$

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Stronger lattice property: If both $\mathcal{C}_A$ and $\mathcal{C}_C$ are increasing and substitutable then lattice operations in Blair's thm are $S_1 \wedge S_2 = \mathcal{C}_A(S_1 \cup S_2)$ and $S_1 \vee S_2 = \mathcal{C}_C(S_1 \cup S_2)$.

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Corollary: There exists a stable marriage scheme in this model.

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Women estimate the strength and the wealth of each man.

Men rank the look of women and the food they cook.

Everyone strives to have (at most) two partners:

- women look for a strong and a wealthy husband and
- man dream about a pretty wife and one that cooks best.

In a marriage scheme, everyone has at most two partners.

Such a scheme is stable if whenever m and w are not married then

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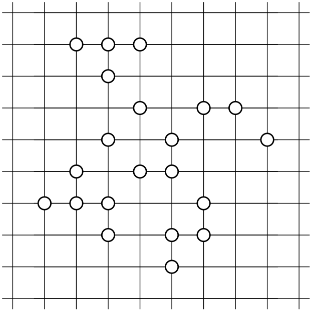
Both $\mathcal{C}_W$ and $\mathcal{C}_M$ are substitutable and PI. So GS works. $\square$

A special case



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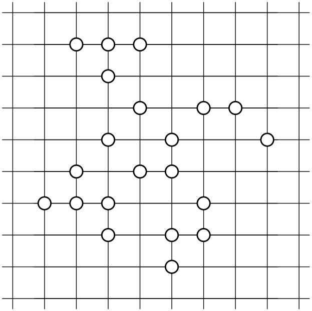

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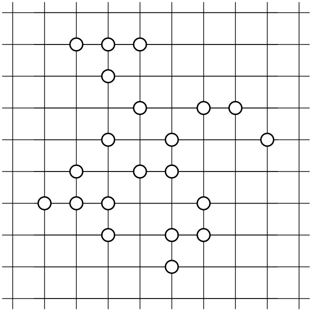

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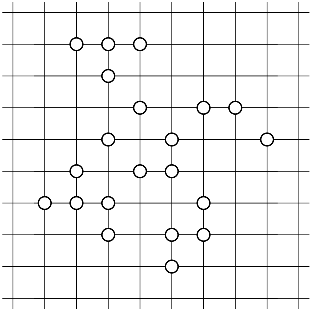

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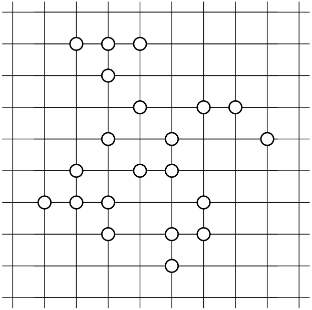

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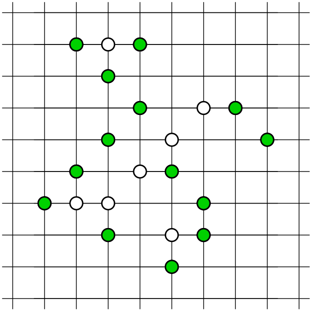

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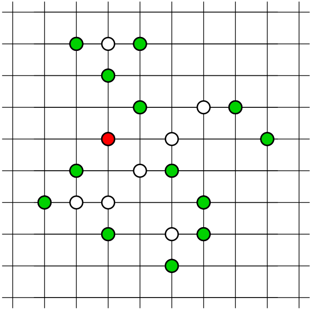

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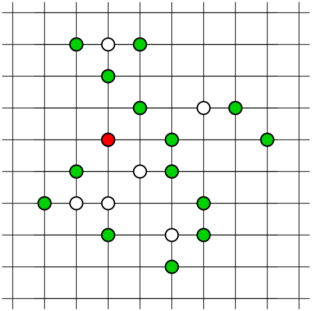

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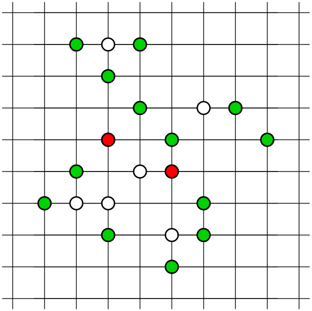

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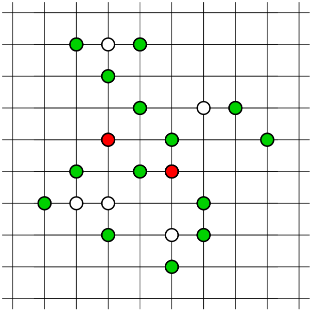

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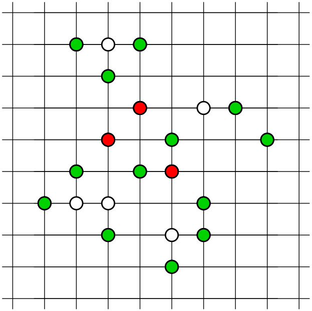

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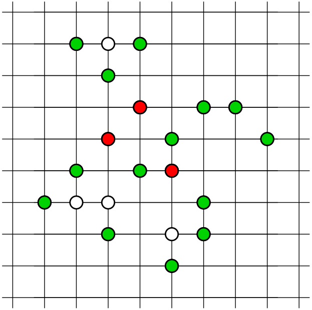

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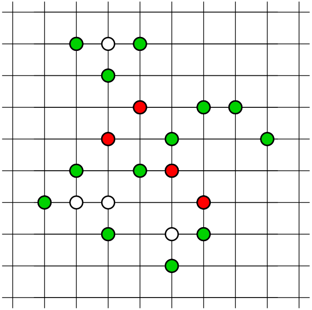

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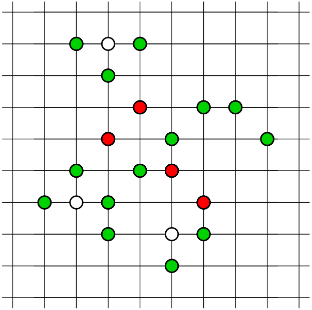

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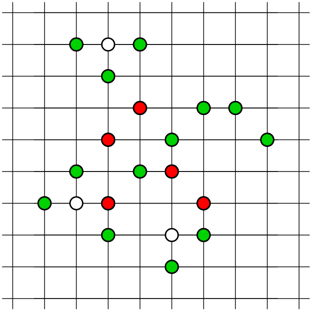

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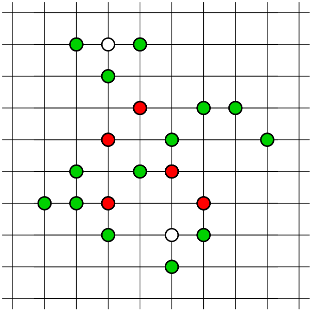

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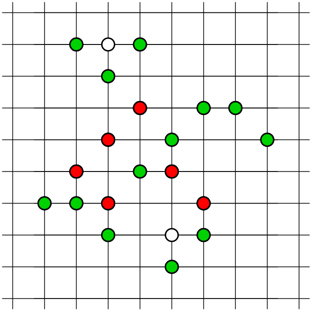

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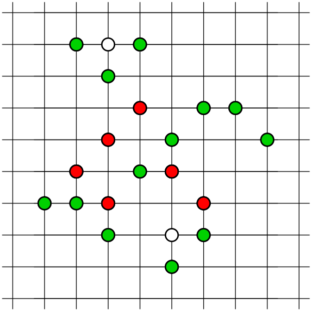

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The man-oriented GS algorithm finds the man-optimal stable solution: the "widest" set of gridpoints. The woman-optimal solution would be the "tallest" such set.

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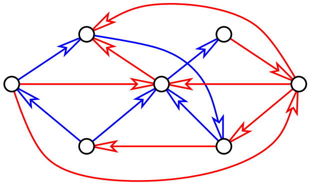

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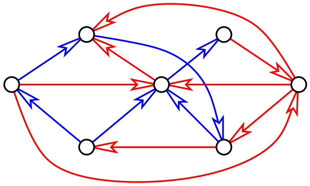

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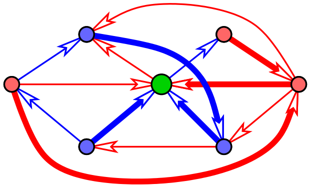

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Stable assignments on many-to-one markets

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NP-completeness: an efficient algorithm for the problem would imply an efficient algorithm for many truly difficult problems.

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Next goal: generalization of Huang's framework.

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Explanation: An applicant might be refused if her admission would imply the violation of some (seemingly independent) lower quota.
Next goal: generalization of Huang's framework.
Main tool: matroid-based choice functions.

A crash course on matroids

Matroid: $\mathcal{M} = (E, \mathcal{I})$ st (1) $\emptyset \in \mathcal{I}$, (2) $A \subseteq B \in \mathcal{I} \Rightarrow A \in \mathcal{I}$,
(3) $A, B \in \mathcal{I}$, $|A| < |B| \Rightarrow \exists b \in B \setminus A : A \cup \{b\} \in \mathcal{I}$.
Examples: (1) Linear matroid (vectors with linear independence)
(2) Graphic matroid (edges of a graph with no cycles)
(3) Trivial matroid ($\mathcal{I} = 2^E$)
(4) Uniform matroid truncation of a trivial matroid
(5) Partition matroid
($E = E_1 \cup E_2 \cup \ldots \cup E_k$ is a partition. $I \in \mathcal{I}$ iff $|I \cap E_i| \leq 1$).
(6) Direct sum of uniform matroids ($E = E_1 \cup E_2 \cup \ldots \cup E_k$ is a partition, $b_1, b_2, \ldots, b_k$ given. $I \in \mathcal{I}$ iff $|I \cap E_i| \leq b_i \forall i$).
Basis: maximal independent set of E (same cardinality)
Rank fn: $rk(A) = \max\{|A'| : A' \subseteq A \text{ independent}\}$.
Span: $sp(A) := \{e \in E : rk(A \cup \{e\}) = rk(A)$.
Greedy prop: maxweight indep set can be constructed greedily deciding on the elements one by one in the order of decr weights.
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"Rural hospitals" Thm: If both $\mathcal{C}_C$ and $\mathcal{C}_A$ are greedy choice fn's then stable assignments have the same span.

The classified stable matching problem

Problem input: Two-sided market between C and A with set E of possible contracts, nested systems $\mathcal{Q}_C, \mathcal{Q}_A \subseteq 2^E$ of common quota sets, $l, u : \mathcal{Q}_A \cup \mathcal{Q}_A \to \mathbb{N}_+$ lower and upper quotas and preferences $\prec_C$ and $\prec_A$ st any common quota set is linearly ordered.

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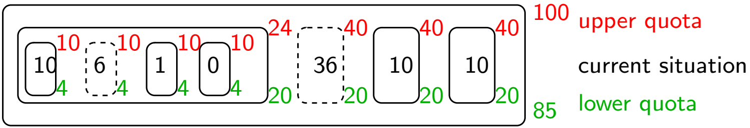
Solution: Application of the choice function framework.

Key question: how do colleges decide on accepted contracts if contracts are coming in the order of preference.

Colleges' choice function



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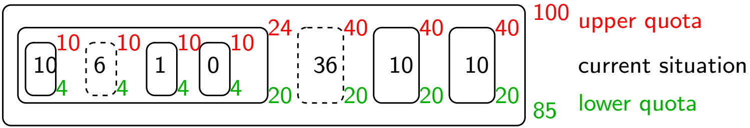

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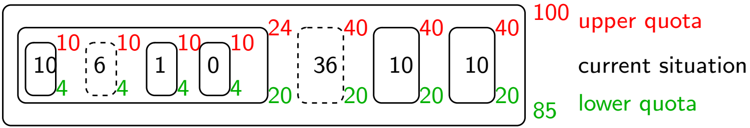

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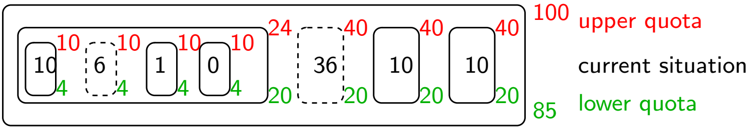

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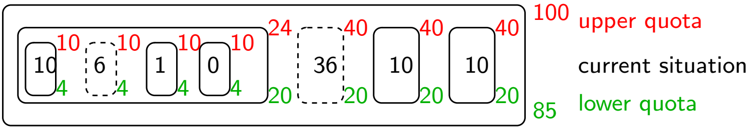

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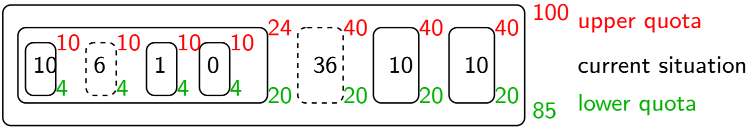

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Trick: As span is always the same, either all $\mathcal{C}_C\mathcal{C}_A$-stable solutions obey the lower quotas or none of them does. So if Gale-Shapley solution violates a lower quota then no stable assignment exists whatsoever. Otherwise GS outputs a solution.

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- **Lesson for Mathematicians**:
  a practical model might motivate a class of interesting matroids

Thank you for the attention!