Generalized stable roommates problems

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Summer School on
Matching Problems, Markets, and Mechanisms
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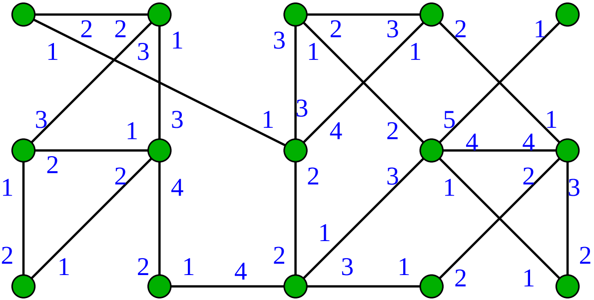
The stable roommates problem



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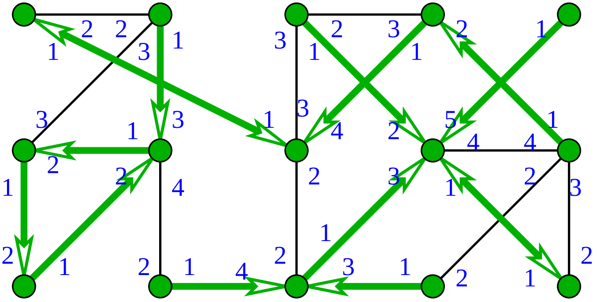

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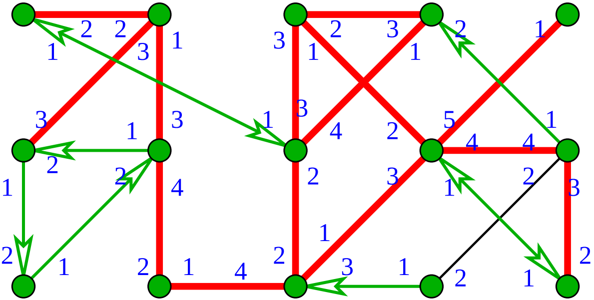

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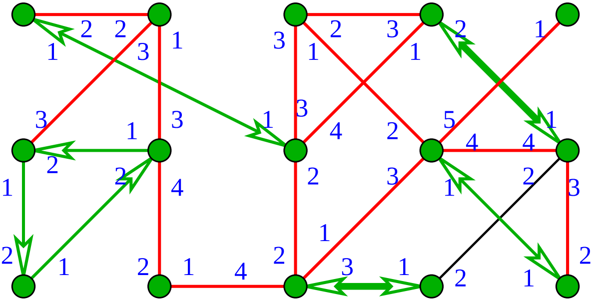

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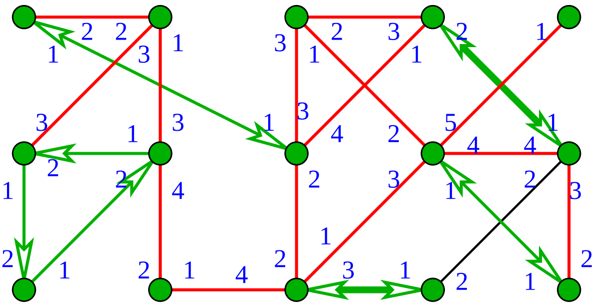

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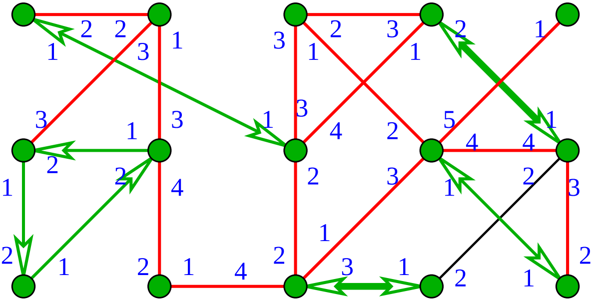

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We shall extend Irving's algorithm to more general situations.

Finding a stable b-matching

Def: Graph $G = (V, E)$ and quota function $b : V \to \mathbb{N}$ is given. A ***b*-matching** is a subset M of e st each vertex v is incident to at most $b(v)$ edges of M. If we also have linear preferences for the vertices then b-matching S is **stable** if it dominates all other edges: if $e = uv \in E \setminus S$ then either u is incident to $b(u)$ edges of S that are all preferred to e or similar holds for v.

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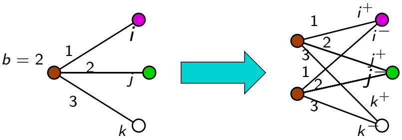

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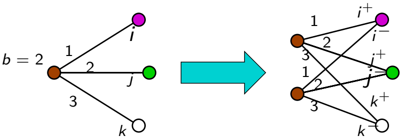

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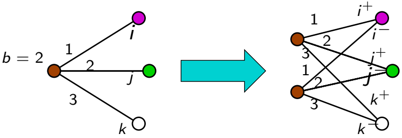

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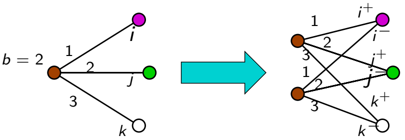

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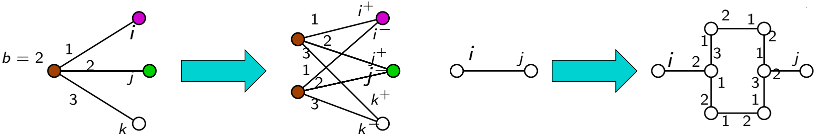

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Solution: A simple construction achieves the mto property.

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- not all stable matchings are killed

until a single stable matching remains or we conclude that no stable matching exists.

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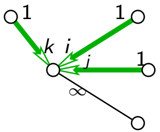

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Def: Edge $e = uv$ is a **1-arc** if e is the first choice of u.

Observation: If v prefers 1-arc $e = uv$ to f then the deletion of f does not change the set of stable matchings.

Basis of Phase 1 of Irving's algorithm.

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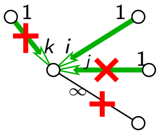

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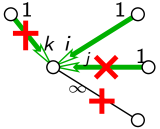

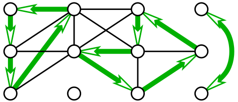

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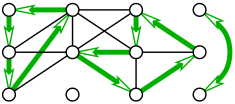
End of Phase 1: If no more GS-type deletion is possible then for each vertex v, the last choice of v is a 1-arc pointing to v, i.e. 1-arcs form vertex-disjoint oriented cycles.

Phase 2



If no more GS-deletion is possible and all 1-arcs are bidirected then we are left with a stable b-matching. Otherwise there is a vertex u incident to at least two edges.

Def: If $e = uv$ is the 2nd choice of v then vu is a **2-arc**.

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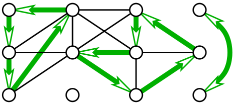

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Corollary: There is a cycle formed alternatingly by 1-arcs and 2-arcs. This is called a rotation.

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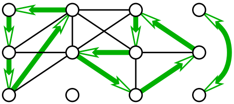

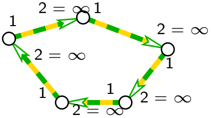

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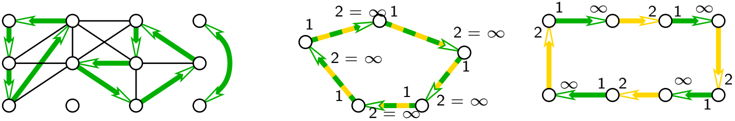

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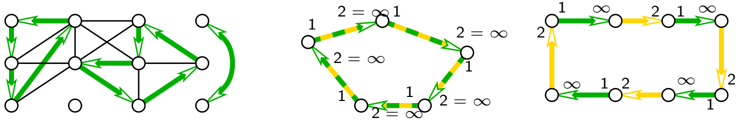

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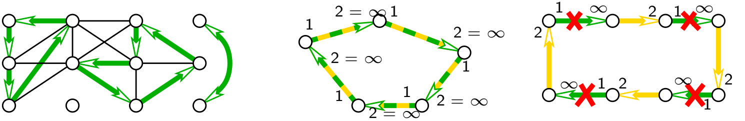

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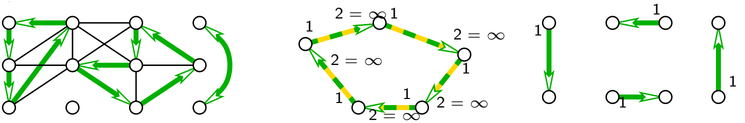

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After eliminating this rotation, reversed 2-arcs become 1-arcs.

(And we may execute further GS-deletions.)

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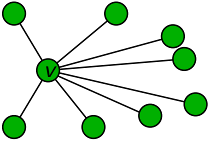

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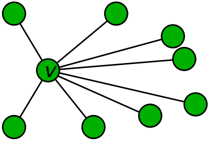

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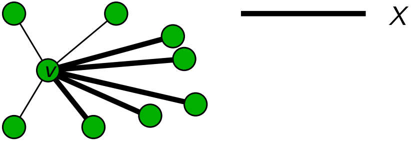

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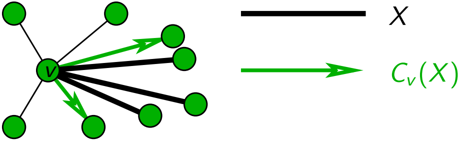

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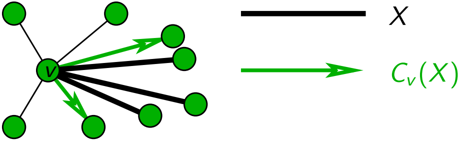

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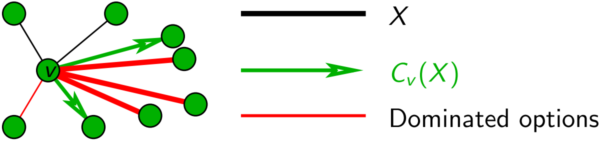

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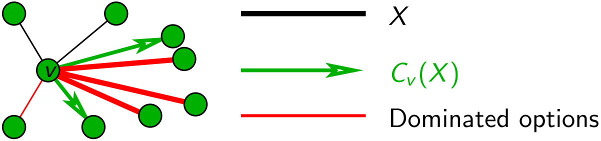

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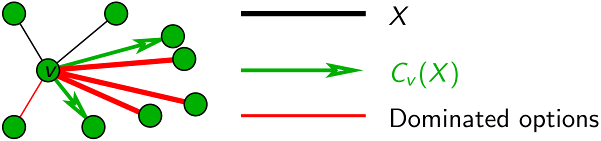

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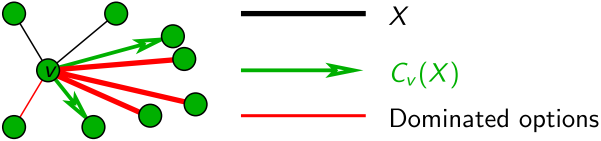

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We assume that agents' choice functions $\mathcal{C}_v$ are substitutable. This means that dominance functions $\mathcal{D}_v$ are monotone:

$$X \subseteq Y \Rightarrow \mathcal{D}_v(X) \subseteq \mathcal{D}_v(Y) \ .$$

(Extra choices do not make an ignored option more attractive.)

Stable matchings with choice functions

So $\mathcal{C}_v(X)$ is the best set of options from X, according to the preference order of v. For the stable b-matching problem, $\mathcal{C}_v(X)$ denotes the best $b(v)$ options of X. We assume that all choice functions $\mathcal{C}_v$ are substitutable.
A stable (b-)matching can be defined as a set S of contracts such that

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Fact: for bipartite graphs, the Gale-Shapley algorithm works.

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The stable partnership problem is given by a graph G and substitutable choice functions $\mathcal{C}_v$ on the stars. Aim: find a stable partnership, i.e. a subset S of $E(G)$ with the above two properties.
Fact: for bipartite graphs, the Gale-Shapley algorithm works.
For nonbipartite graphs, we can solve only a special case: we assume that each choice function $\mathcal{C}_v$ is increasing, i.e.

$$X \subseteq Y \Rightarrow |\mathcal{C}_v(X)| \leq |\mathcal{C}_v(Y)| \ .$$

(Greater choice set means more choices selected.)

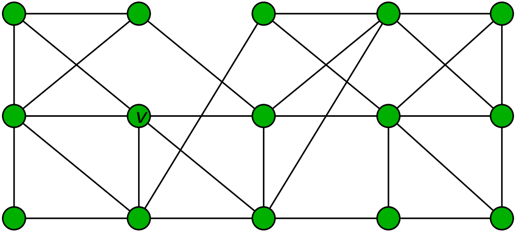
Finding a stable partnership

Generalization of Irving's algorithm: we keep on deleting edges such that

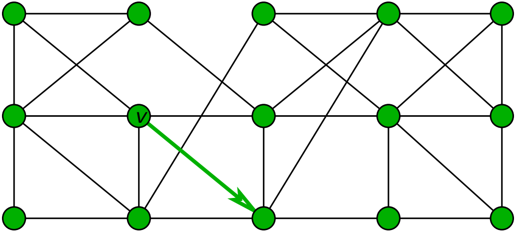
- no new stable partnership is created
- not all stable partnerships are killed

until a single stable partnership remains.
For an ordinary stable roommates problem, the extended algorithm is doing the same as Irving's.

1st phase of the algorithm



1st phase of the algorithm



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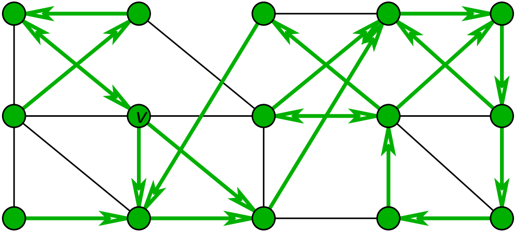

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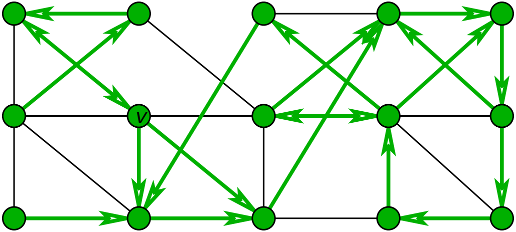
Proposal step: Find all 1-arcs.

1st phase of the algorithm



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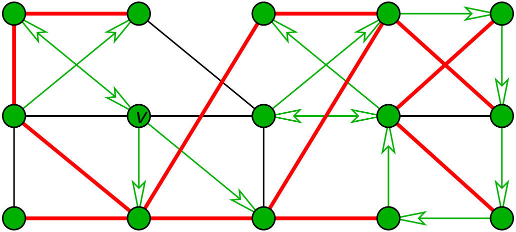
Proposal step: Find all 1-arcs.

The set of stable partnerships does not change.

1st phase of the algorithm



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The set of stable partnerships does not change.

Refusal step: If X is the set of 1-arcs pointing to u, delete $\mathcal{D}_u(X)$.

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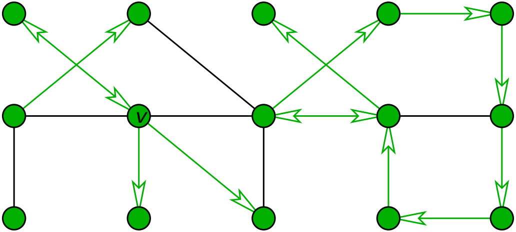

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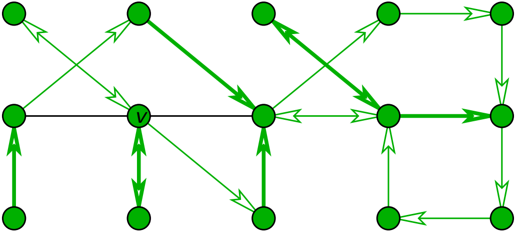

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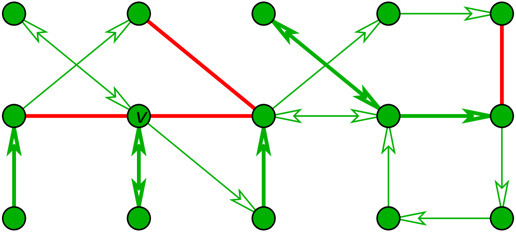
Execute proposal and refusal steps alternatingly

1st phase of the algorithm



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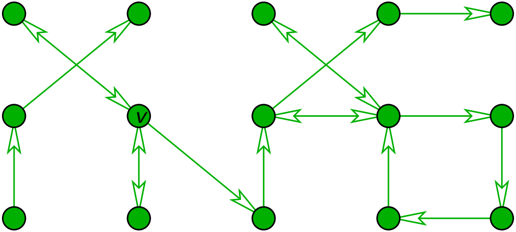

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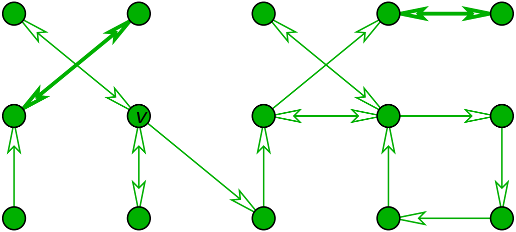

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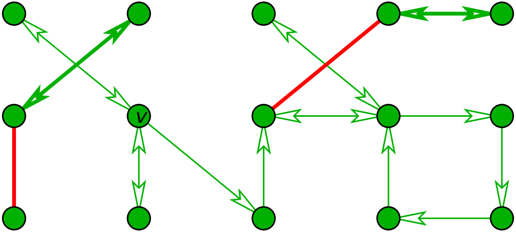

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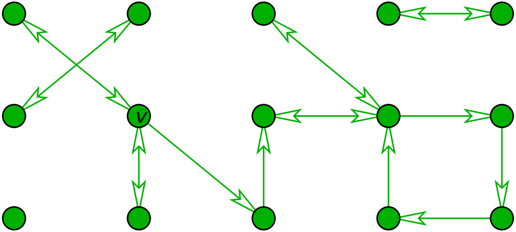

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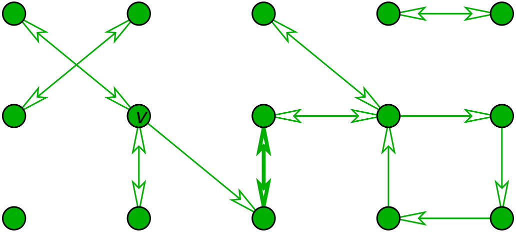

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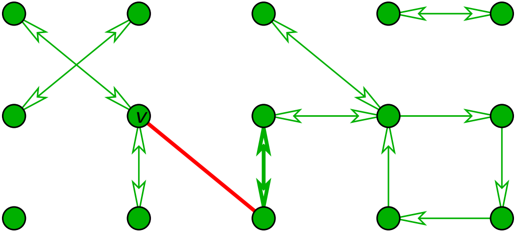

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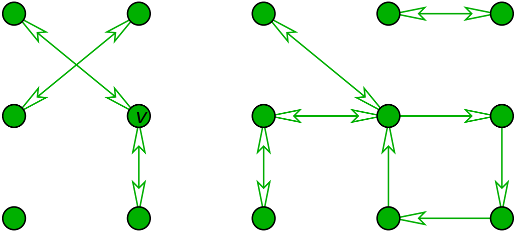

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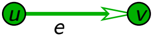
The set of stable partnerships does not change.

Execute proposal and refusal steps alternatingly until no further step is possible. Now 1-arcs form an Eulerian graph.

If all 1-arcs are bidirected then it is the only stable partnership.

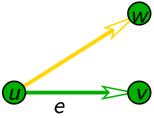
Otherwise we move on to the 2nd phase of the algorithm.

Finding a rotation from replacements



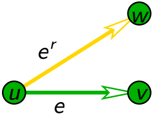
Let $e = uv$ be a 1-arc.

Finding a rotation from replacements



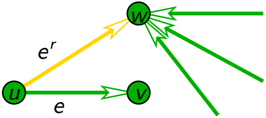
Let $e = uv$ be a 1-arc. The replacement of e is the contract that u selects instead of e if e is not available any more,

Finding a rotation from replacements



Let $e = uv$ be a 1-arc. The replacement of e is the contract that u selects instead of e if e is not available any more, that is, $e^r = C_u(E(u) \setminus \{e\}) \setminus C_u(E(u))$.

Finding a rotation from replacements



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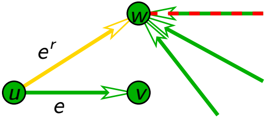

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$A^- \cup \{e^r\}$ dominates a unique 1-arc e_r^r.

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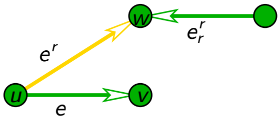

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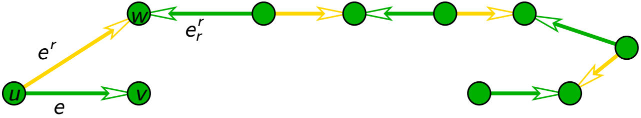

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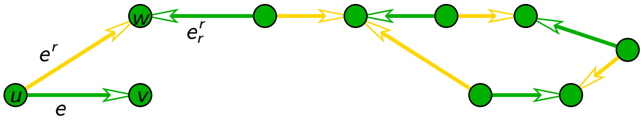
Alternating sequence

$$e, e^r, e_r^r, (e_r^r)^r, (e_r^r)_r^r, \ldots$$

of 1-arcs and replacements

Finding a rotation from replacements



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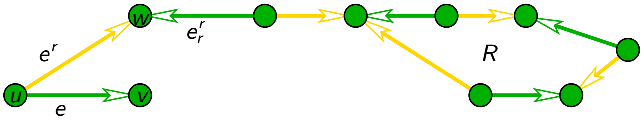
Finding a rotation from replacements



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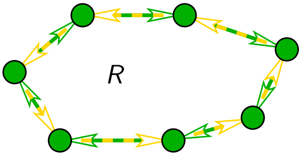
of 1-arcs and replacements sooner or later repeats a 1-arc.
Hence we find a rotation R.

Rotation elimination

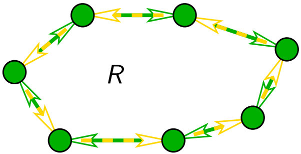
Case 1. 1-arcs and their replacements in R form identical sets.

Rotation elimination



Case 1. 1-arcs and their replacements in R form identical sets.
Rotation R is an odd cycle, and the algorithm stops:

Rotation elimination



Case 1. 1-arcs and their replacements in R form identical sets.
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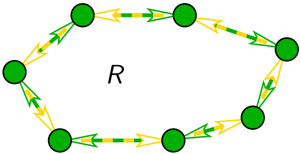
No stable partnership exists.

Rotation elimination



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Rotation R is an odd cycle, and the algorithm stops:

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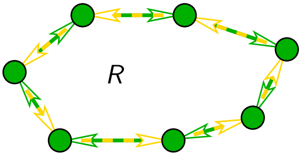
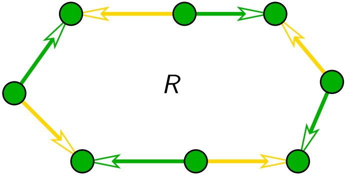
Case 2. 1-arcs and their replacements in R are distinct.

Rotation elimination





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Case 2. 1-arcs and their replacements in R are distinct.
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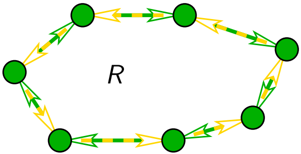
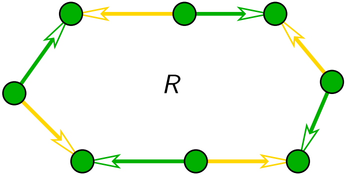
Rotation elimination





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Case 2. 1-arcs and their replacements in R are distinct.
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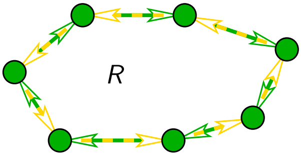
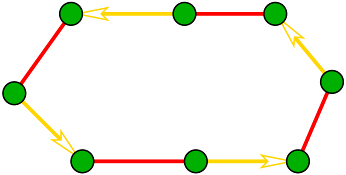
We eliminate the rotation:

Rotation elimination





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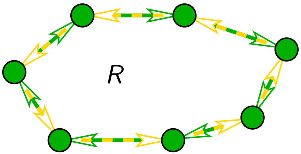
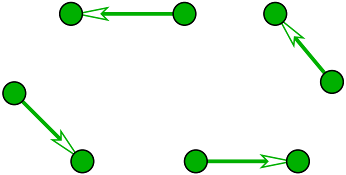
We eliminate the rotation:

- Delete all 1-arcs of the rotation

Rotation elimination





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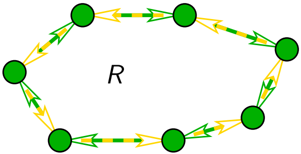
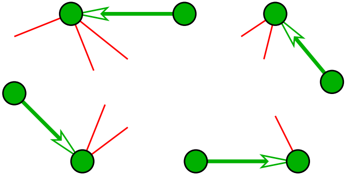
We eliminate the rotation:

- Delete all 1-arcs of the rotation
- Replacements become 1-arcs.

Rotation elimination





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We eliminate the rotation:

- Delete all 1-arcs of the rotation
- Replacements become 1-arcs.
- Execute a refusal step at each terminal of the new 1-arcs.

2nd phase of the algorithm

Theorem

After a rotation elimination, no new stable partnership is created and not **all** *stable partnerships are killed.*

How does the algorithm terminate?

Theorem

If there are no more rotations then all edges are bidirected 1*-arcs, hence the graph itself is a stable partnership.*

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Complexity?

Theorem

The generalization of Irving's algorithm needs $O(n+m)$ $\mathcal{C}$*-calls and* $O(n+m)$ $\mathcal{D}$*-calls.*

Thank you for the attention!