University of Glasgow

Summer School on Matching Problems, Markets and Mechanisms

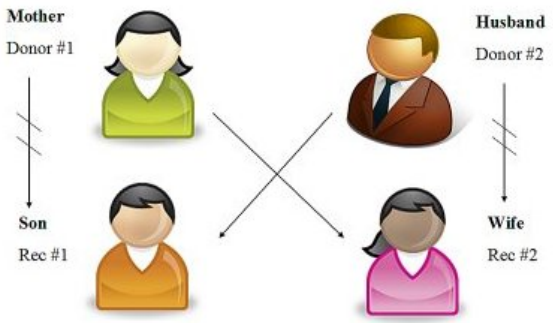
David Manlove

Outline

1. The Hospitals / Residents problem and its variants

2. The House Allocation problem

3. Kidney exchange



The House Allocation Problem

with applications to reviewer assignment

aka bipartite matching with one-sided preferences

House Allocation problem (HA)

- Set of applicants $A=\{a_1, a_2, \ldots, a_r\}$
- Set of houses $H=\{h_1, h_2, \ldots, h_s\}$

- Each applicant a_i has an *acceptable* set of houses $A_i \subseteq H$
- a_i ranks A_i in strict order of preference
- Houses do not have preferences over applicants

- Example:

$a_1 : h_2 \; h_1$
$a_2 : h_3 \; h_4 \; h_2$
$a_3 : h_4 \; h_3$
$a_4 : h_1 \; h_4$

a_1 finds h_1 and h_2 acceptable

a_3 prefers h_4 to h_3

- Let $n=r+s$ and let m=total length of preference lists
- "Applicants" could be reviewers and "houses" could be papers

Applications of HA

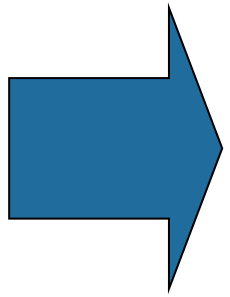
- Campus housing allocation in the US and Israel
  - Carnegie-Mellon, Duke, Michigan, Northwestern, Technion
  - [Chen and Sönmez, 2002; Perach, Polak and Rothblum, 2008]
- Allocating families to government-subsidised housing in China
  - [Yuan, 1996]
- Allocating students to projects and elective courses
  - Schools of Computing Science and Medicine, University of Glasgow
- DVD rental by post
  - [Abraham, Chen, Kumar and Mirrokni, 2006]
- Conference paper reviewer assignment (Easychair etc.)
  - [Garg, Kavitha, Kumar, Mehlhorn and Mestre, 2010]

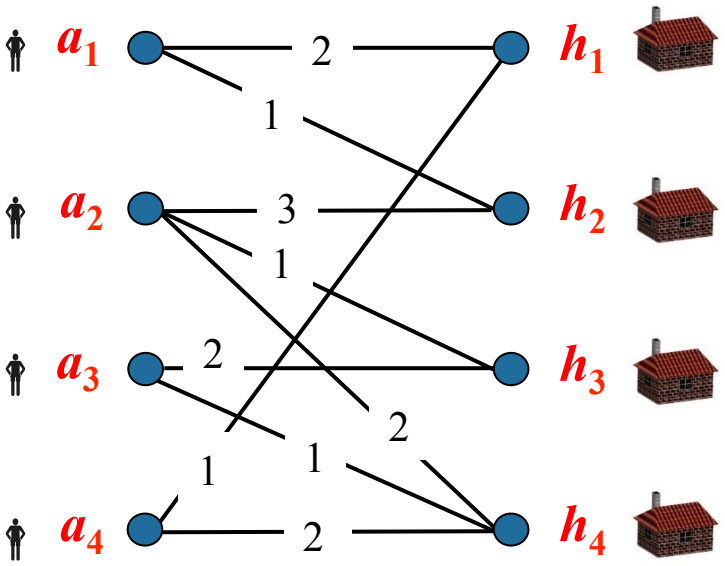
The underlying graph

- Weighted bipartite graph $G=(V,E)$
  - Vertex set $V=A \cup H$
  - Edge set: $\{a_i, h_j\} \in E$ if and only if a_i finds h_j acceptable
  - Weight of edge $\{a_i, h_j\}$ is rank of h_j in a_i's preference list

- Example:

  $a_1 : h_2\ h_1$
  $a_2 : h_3\ h_4\ h_2$
  $a_3 : h_4\ h_3$
  $a_4 : h_1\ h_4$





Definition of a matching

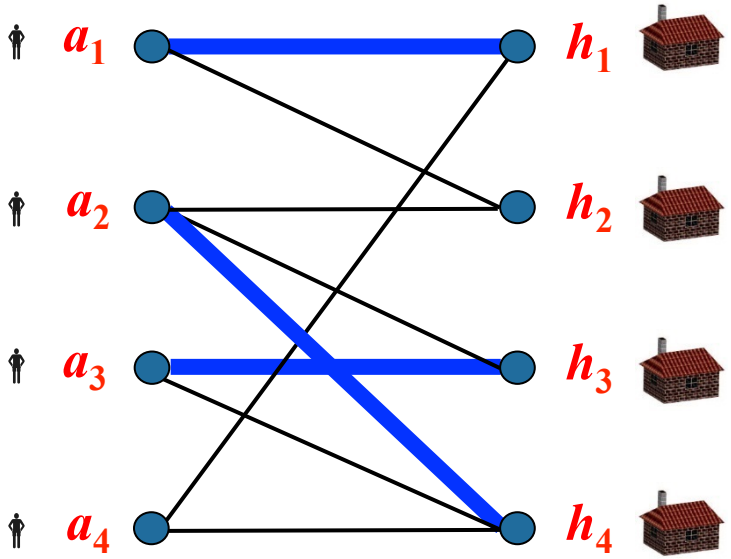
- A *matching* M is a subset of $A \times H$ such that:
  - if $(a_i, h_j) \in M$ then a_i finds h_j acceptable
  - each applicant belongs to at most one pair in M
  - each house belongs to at most one pair in M

- Example:

$a_1 : h_2\ h_1$
$a_2 : h_3\ h_4\ h_2$
$a_3 : h_4\ h_3$
$a_4 : h_1\ h_4$

$M(a_1)=h_1$

$M=\{(a_1, h_1), (a_2, h_4), (a_3, h_3)\}$



Maximum matchings

- In many applications we wish to match as many applicants as possible
- A *maximum matching* is a matching with largest size
- Example

$a_1 : h_2\ \ h_1$
$a_2 : h_3\ \ h_4\ \ h_2$
$a_3 : h_4\ \ h_3$
$a_4 : h_1\ \ h_4$

$a_1 : h_2\ \ h_1$
$a_2 : h_3\ \ h_4\ \ h_2$
$a_3 : h_4\ \ h_3$
$a_4 : h_1\ \ h_4$

- Some maximum matchings are “better” than others!
- Applicant a *prefers* matching M to matching M' if
  - a is matched in M but not in M', or
  - a is matched in both M and M' and prefers $M(a)$ to $M'(a)$

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Tutorial Outline

2.1: Pareto optimal matchings

2.2: Popular matchings

2.3: Profile-based optimal matchings

2.1: Pareto optimal matchings

2.2: Popular matchings

2.3: Profile-based optimal matchings

Pareto optimal matchings

- A matching M_1 is *Pareto optimal* if there is no matching M_2 such that:
  1. Some applicant prefers M_2 to M_1
  2. No applicant prefers M_1 to M_2

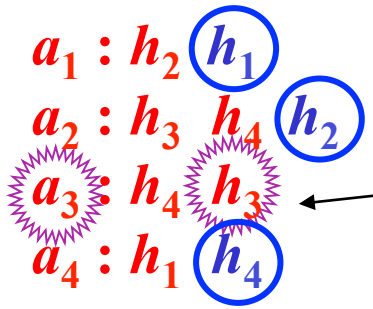
- Example

- M_1 is not Pareto optimal since a_1 and a_2 could swap houses – each would be better off!

- M_2 is Pareto optimal

Testing for Pareto optimality

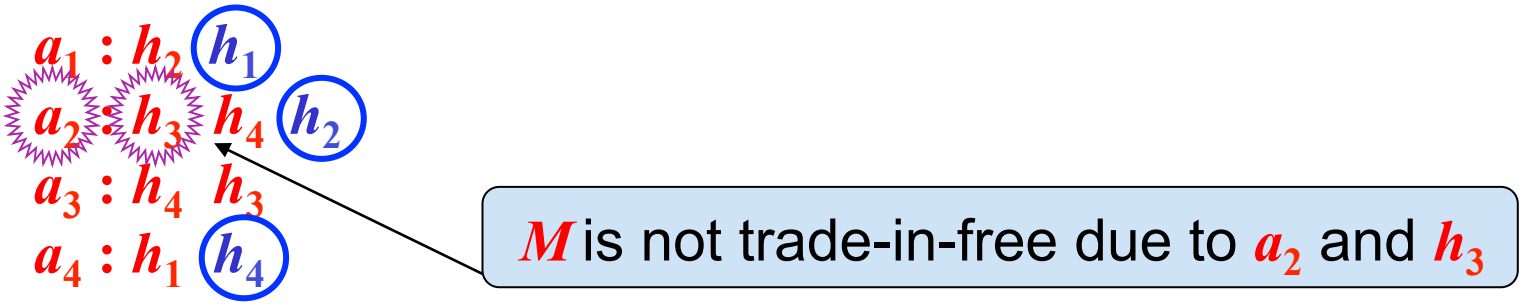
- A matching M is *maximal* if there is no applicant a and house h, each unmatched in M, such that a finds h acceptable



M is not maximal due to a_3 and h_3

Testing for Pareto optimality

- A matching M is *trade-in-free* if there is no matched applicant a and unmatched house h such that a prefers h to $M(a)$

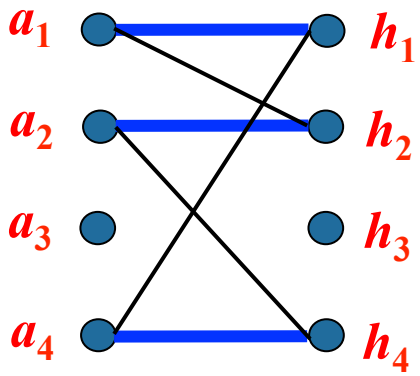


Testing for Pareto optimality

- A matching M is *coalition-free* if there is no *coalition*, i.e. a sequence of matched applicants $\langle a_0, a_1, \ldots, a_{r-1} \rangle$ such that a_i prefers $M(a_{i+1})$ to $M(a_i)$ $(0 \leq i \leq r\text{-}1)$

$a_1 : \underline{h_2}\ (h_1)$
$a_2 : h_3\ \underline{h_4}\ (h_2)$
$a_3 : h_4\ h_3$
$a_4 : \underline{h_1}\ (h_4)$

M is not coalition-free due to $\langle a_1, a_2, a_4 \rangle$



Testing for Pareto optimality

- A matching M is *maximal* if there is no applicant a and house h, each unmatched in M, such that a finds h acceptable
- A matching M is *trade-in-free* if there is no matched applicant a and unmatched house h such that a prefers h to $M(a)$
- A matching M is *coalition-free* if there is no *coalition*, i.e. a sequence of matched applicants $\langle a_0, a_1, \ldots, a_{r-1} \rangle$ such that a_i prefers $M(a_{i+1})$ to $M(a_i)$ $(0 \leq i \leq r\text{-}1)$

$a_1 : h_2$ (h_1)
$a_2 : h_3$ h_4 (h_2)
$a_3 : h_4$ h_3
$a_4 : h_1$ (h_4)

- Proposition: M is Pareto optimal if and only if M is maximal, trade-in-free and coalition-free

The envy graph

- Straightforward to check a given matching M for the maximality and trade-in-free properties in $O(m)$ time

- To check for the existence of a coalition:
  - Form the envy graph of M, denoted by $G(M)$
  - Vertex for each matched applicant
  - Edge from a_i to a_j if and only if a_i prefers $M(a_j)$ to $M(a_i)$

- Example

$a_1 : \underline{h_2}\ (h_1)$
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$a_3 : h_4\ h_3$
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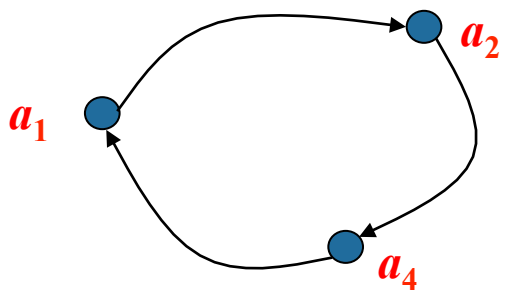

- M admits a coalition if and only if $G(M)$ has a directed cycle

- Proposition: we may check whether a given matching M is Pareto optimal in $O(m)$ time

Finding a Pareto optimal matching

- Simple greedy algorithm, called Algorithm Greedy-POM
  - referred to as a “serial dictatorship mechanism” by economists

```
for each applicant a in turn
    if a has an unmatched house on his list
        match a to the most-preferred such house;
    else
        report a as unmatched;
```

- Example

$a_1 : h_1\ h_2\ h_3$
$a_2 : h_1\ h_2$
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$M_1=\{(a_1,h_1), (a_2,h_2)\}$

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$M_2=\{(a_1,h_3), (a_2,h_2), (a_3,h_1)\}$

$a_1 : h_1\ h_2\ h_3$
$a_2 : h_1\ h_2$
$a_3 : h_1\ h_2$

- Proposition: Algorithm Greedy-POM constructs a Pareto optimal matching in $O(m)$ time

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Finding a maximum Pareto optimal matching

- Take underlying weighted bipartite graph G
  - The weight of a matching M in G is the sum of the weights of the edges contained in M

$$wt(M) = \sum_{a \in A_M} rank_a(M(a))$$

    - where A_M is the set of applicants who are matched in M
    - $rank_a(h)$ is the rank of h in a's preference list
  - Find a maximum cardinality minimum weight matching M in G
  - M is Pareto optimal
  - For if not there exists a matching M' such that
    - $rank_a(M'(a)) \leq rank_a(M(a))$ for all $a \in A_M$
    - $rank_a(M'(a)) < rank_a(M(a))$ for some $a \in A_M$
  - so M' contradicts the minimum weight property of M
- A maximum cardinality minimum weight matching may be found in $O(\sqrt{n}m \log n)$ time
  - [Gabow and Tarjan, 1989]

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A faster algorithm

- Three-phase algorithm with $O(\sqrt{nm})$ overall complexity
  - [Abraham, Cechlárová, M and Mehlhorn, 2004]
- Phase 1 – $O(\sqrt{nm})$ time
  - Find a maximum matching in G
  - Classical $O(\sqrt{nm})$ augmenting path algorithm
    - [Hopcroft and Karp, 1973]
- Phase 2 – $O(m)$ time
  - Enforce trade-in-free property
- Phase 3 – $O(m)$ time
  - Enforce coalition-free property
  - Extension of Gale's Top-Trading Cycles (TTC) algorithm
    - [Shapley and Scarf, 1974]

Phase 1

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$

$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$

$a_4 : h_3\ h_5\ h_4$

$a_5 : h_4\ h_3\ h_5$

$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ h_9$

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Phase 1

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$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ (h_2)$
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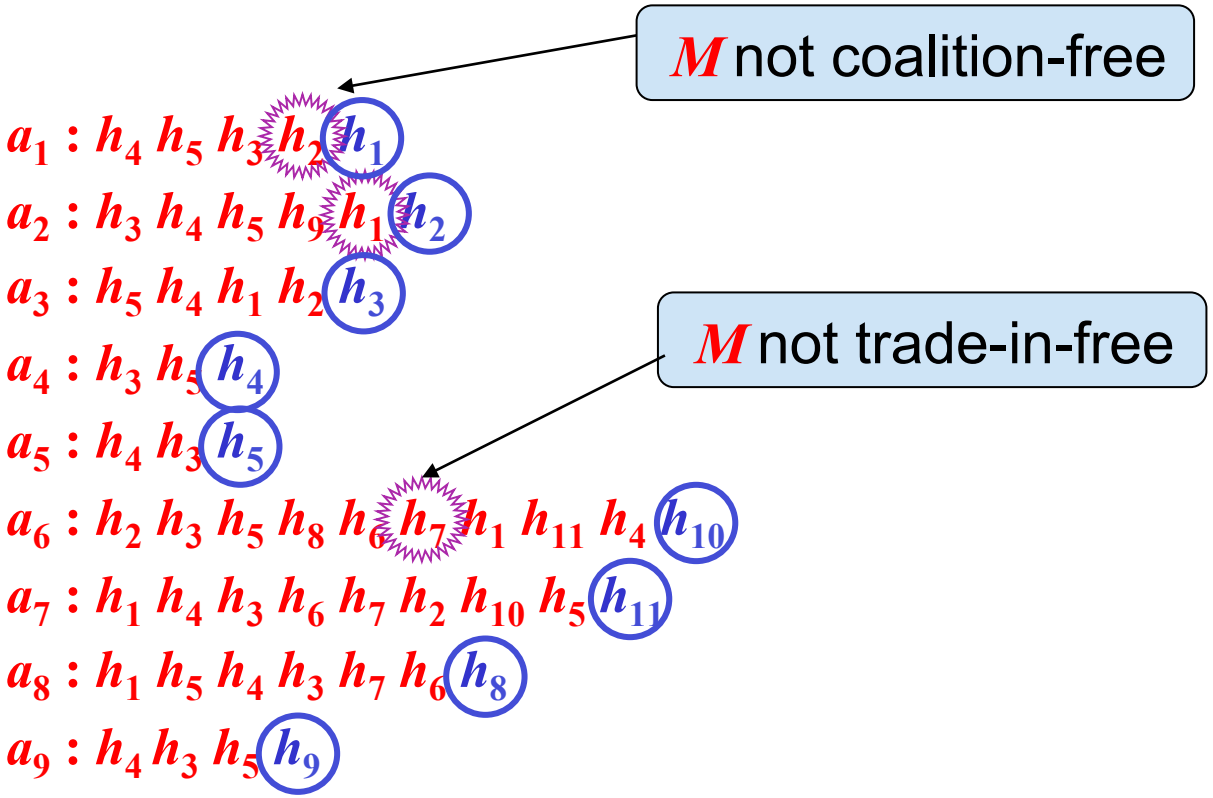
$a_9 : h_4\ h_3\ h_5\ (h_9)$

- Maximum matching M in G has size 9
- M must be maximal
- No guarantee that M is trade-in-free or coalition-free

Phase 1



- Maximum matching M in G has size 9
- M must be maximal
- No guarantee that M is trade-in-free or coalition-free

Phase 2 description

- For each house h, maintain a list L_h, initially containing those pairs (a, r) such that:
  - a is a matched applicant who prefers h to $M(a)$
  - r is the rank of h in a's list
- Maintain a stack S of unmatched houses h whose list L_h is nonempty
- Each matched applicant a maintains a pointer $curr_a$ to the rank of $M(a)$

- Example

$a_1 : h_4\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : h_5\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : h_3\ h_5\ (h_4)$
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$L_{h_6} = \langle (a_6, 5), (a_7, 4), (a_8, 6) \rangle$
$L_{h_7} = \langle (a_6, 6), (a_7, 5), (a_8, 5) \rangle$
$L_{h_8} = \langle (a_6, 4) \rangle$
$L_{h_{10}} = \langle (a_7, 7) \rangle$
$L_{h_{11}} = \langle (a_6, 8) \rangle$
$S = \langle h_6, h_7 \rangle$
$curr_{a_6} = 10$
$curr_{a_7} = 9$

Phase 2 description

```
while S is nonempty
    pop a house h from S;
    remove the first pair (a,r) from the head of L_h;
    if r < curr_a // a prefers h to M(a)
        let h' = M(a);
        remove (a,h') from M and add (a,h) to M;
        curr_a := r;
        h := h';
    push h onto the stack if L_h is nonempty;
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$a_3 : h_5\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : h_3\ h_5\ (h_4)$
$a_5 : h_4\ h_3\ (h_5)$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ (h_7)\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ (h_{11})$
$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : h_4\ h_3\ h_5\ (h_9)$

$L_{h_6} = \langle (a_6, 5), (a_7, 4), (a_8, 6) \rangle$
$L_{h_7} = \langle (a_7, 5), (a_8, 5) \rangle$
$L_{h_8} = \langle (a_6, 4) \rangle$
$L_{h_{10}} = \langle (a_7, 7) \rangle$
$L_{h_{11}} = \langle (a_6, 8) \rangle$
$S = \langle h_6, h_{10} \rangle$
$curr_{a_6} = 6$
$curr_{a_7} = 9$

Phase 2 example

```
while S is nonempty
    pop a house h from S;
    remove the first pair (a,r) from the head of L_h;
    if r < curr_a // a prefers h to M(a)
        let h' = M(a);
        remove (a,h') from M and add (a,h) to M;
        curr_a := h;
        h := h';
    push h onto the stack if L_h is nonempty;
```

$a_1 : h_4\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : h_5\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : h_3\ h_5\ (h_4)$
$a_5 : h_4\ h_3\ (h_5)$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ (h_7)\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ (h_{10})\ h_5\ h_{11}$
$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : h_4\ h_3\ h_5\ (h_9)$

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$L_{h_{10}} = \langle\ \rangle$
$L_{h_{11}} = \langle (a_6, 8) \rangle$
$S = \langle h_6, h_{11} \rangle$
$curr_{a_6} = 6$
$curr_{a_7} = 7$

Phase 2 example

```
while S is nonempty
    pop a house h from S;
    remove the first pair (a,r) from the head of L_h;
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$a_1 : h_4\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : h_5\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : h_3\ h_5\ (h_4)$
$a_5 : h_4\ h_3\ (h_5)$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ (h_7)\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ (h_{10})\ h_5\ h_{11}$
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$a_9 : h_4\ h_3\ h_5\ (h_9)$

$L_{h_6}=\langle(a_6, 5), (a_7, 4), (a_8, 6)\rangle$
$L_{h_7}=\langle(a_7, 5), (a_8, 5)\rangle$
$L_{h_8}=\langle(a_6, 4)\rangle$
$L_{h_{10}}=\langle\ \rangle$
$L_{h_{11}}=\langle(a_6, 8)\rangle$
$S=\langle h_6, h_{11}\rangle$
$curr_{a_6}=6$
$curr_{a_7}=7$

Phase 2 example

```
while S is nonempty
    pop a house h from S;
    remove the first pair (a,r) from the head of L_h;
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$a_1 : h_4\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : h_5\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : h_3\ h_5\ (h_4)$
$a_5 : h_4\ h_3\ (h_5)$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ (h_7)\ h_1\ h_{11}\ h_4\ h_{10}$
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$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
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$L_{h_6} = \langle (a_6, 5), (a_7, 4), (a_8, 6) \rangle$
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$L_{h_{10}} = \langle\ \rangle$
$L_{h_{11}} = \langle\ \rangle$
$S = \langle h_6 \rangle$
$curr_{a_6} = 6$
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Phase 2 example

```
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Phase 2 example

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$a_3 : h_5\ h_4\ h_1\ h_2\ (h_3)$
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```
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$L_{h_{10}} = \langle\ \rangle$
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$curr_{a_7} = 5$

Phase 2 example

```
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$a_4 : h_3\ h_5\ (h_4)$
$a_5 : h_4\ h_3\ (h_5)$
$a_6 : h_2\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
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$L_{h_6} = \langle (a_7, 4), (a_8, 6) \rangle$
$L_{h_7} = \langle (a_8, 5) \rangle$
$L_{h_8} = \langle (a_6, 4) \rangle$
$L_{h_{10}} = \langle\ \rangle$
$L_{h_{11}} = \langle\ \rangle$
$S = \langle\ \rangle$
$curr_{a_6} = 5$
$curr_{a_7} = 5$

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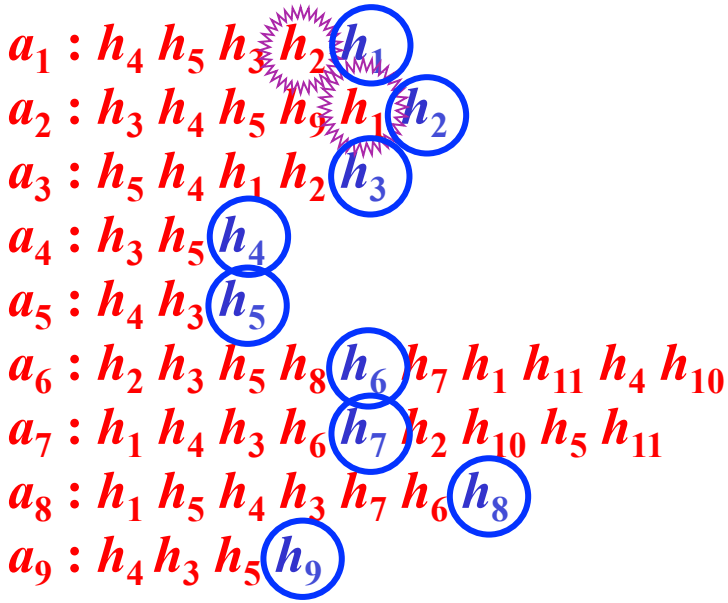
Phase 2 termination

- Once Phase 2 terminates, matching is trade-in-free
- Data structures may be initialised in $O(m)$ time
- Main loop takes $O(m)$ time overall
- ∴ Phase 2 is $O(m)$
- Coalitions may remain…

Phase 2 termination

- Once Phase 2 terminates, matching is trade-in-free
- Data structures may be initialised in $O(m)$ time
- Main loop takes $O(m)$ time overall
- ∴ Phase 2 is $O(m)$
- Coalitions may remain…

- Example



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Phase 3 description

- Once Phase 2 terminates, matching is trade-in-free
- Still need to eliminate potential coalitions
- Repeatedly finding and eliminating coalitions takes $O(m^2)$ time
  - Cycle detection in $G(M)$ takes $O(m)$ time
  - $O(m)$ coalitions in the worst case
- Faster method: extension of TTC algorithm
  - An applicant matched to his/her first-choice house cannot be in a coalition
    - Such an applicant can be removed from consideration
  - Houses matched to such applicants are no longer exchangeable
    - Such a house can be removed from consideration
  - This rule can be recursively applied until either
    - No applicant remains (matching is coalition-free)
    - A coalition exists, which can be found and removed

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Phase 3 description

- Build a path P of applicants (represented by a stack)
- Each house is initially unlabelled
- Each applicant a has a pointer $p(a)$ pointing to $M(a)$ or the first unlabelled house on a's preference list (whichever comes first)
- Keep a counter $c(a)$ for each applicant a (initially $c(a)=0$)
  - This represents the number of times a appears on the stack
- Outer loop iterates over each matched applicant a such that $p(a) \neq M(a)$
- Initialise P to contain applicant a
- Inner loop iterates while P is nonempty
  - Pop an applicant a' from P
  - If $c(a')=2$ we have a coalition (CYCLE)
    - Remove by popping the stack and label the houses involved
  - Else if $p(a')=M(a')$ we reach a dead end (BACKTRACK)
    - Label $M(a')$
  - Else add a'' where $p(a')=M(a'')$ to the path (EXTEND)
    - Push a' and a'' onto the stack
    - Increment $c(a'')$

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : \underline{h_3}\ h_5\ (h_4)$
$a_5 : \underline{h_4}\ h_3\ (h_5)$
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ (h_7)\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : \underline{h_4}\ h_3\ h_5\ (h_9)$

applicant	counter	house	label
a_1	1	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	

a_1 

$P=\langle a_1 \rangle$

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : \underline{h_3}\ h_5\ (h_4)$
$a_5 : \underline{h_4}\ h_3\ (h_5)$
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
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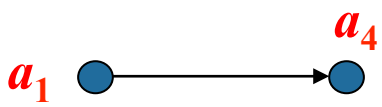
$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : \underline{h_4}\ h_3\ h_5\ (h_9)$

applicant	counter	house	label
a_1	1	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	1	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



EXTEND

$P = \langle a_1, a_4 \rangle$

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : \underline{h_3}\ h_5\ (h_4)$
$a_5 : \underline{h_4}\ h_3\ (h_5)$
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
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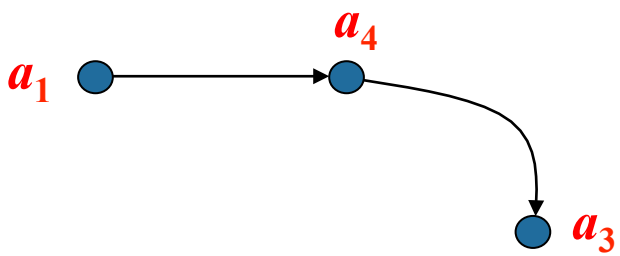
applicant	counter	house	label
a_1	1	h_1	
a_2	0	h_2	
a_3	1	h_3	
a_4	1	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



EXTEND

$P = \langle a_1, a_4, a_3 \rangle$

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ (h_3)$
$a_4 : \underline{h_3}\ h_5\ (h_4)$
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a_4	1	h_4	
a_5	1	h_5	
a_6	0	h_6	
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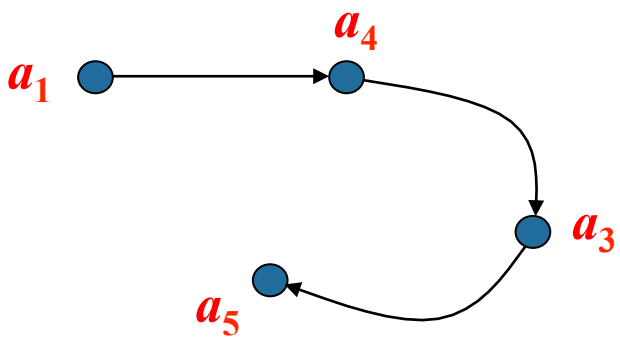

$P = \langle a_1, a_4, a_3, a_5 \rangle$

EXTEND

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
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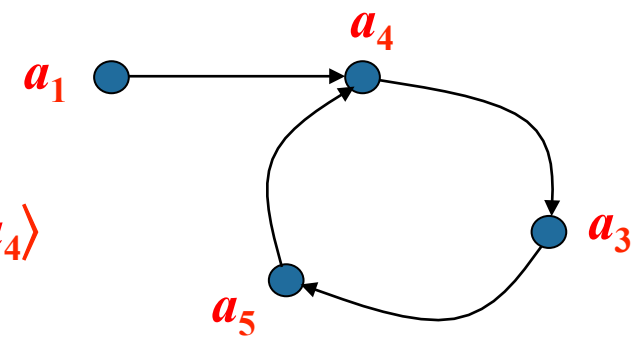
a_1	1	h_1	
a_2	0	h_2	
a_3	1	h_3	
a_4	2	h_4	
a_5	1	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



$P = \langle a_1, a_4, a_3, a_5, a_4 \rangle$

EXTEND

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
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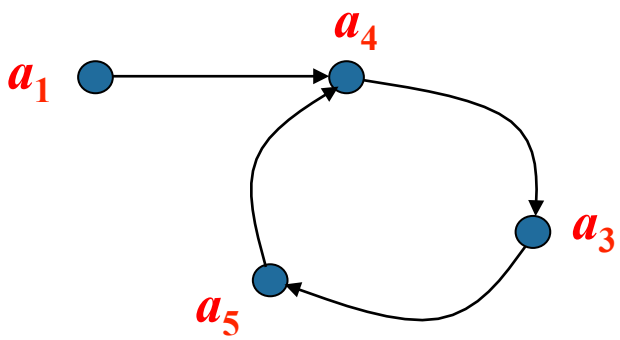
$a_5 : \underline{h_4}\ h_3\ (h_5)$
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ (h_7)\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : \underline{h_4}\ h_3\ h_5\ (h_9)$

applicant	counter	house	label
a_1	1	h_1	
a_2	0	h_2	
a_3	1	h_3	
a_4	2	h_4	
a_5	1	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



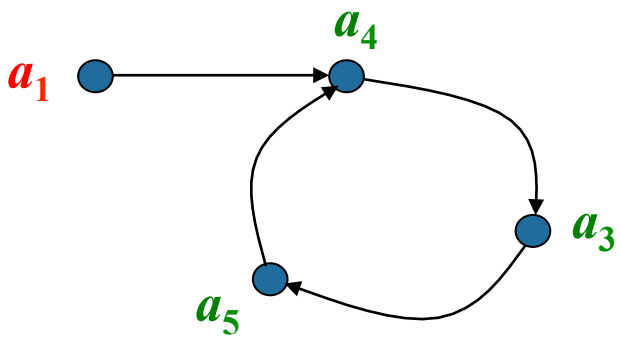
$P = \langle a_1, a_4, a_3, a_5 \rangle$

CYCLE

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : (\underline{h_5})\ h_4\ h_1\ h_2\ h_3$
$a_4 : (\underline{h_3})\ h_5\ h_4$
$a_5 : (\underline{h_4})\ h_3\ h_5$
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ (h_7)\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : \underline{h_4}\ h_3\ h_5\ (h_9)$

applicant	counter	house	label
a_1	1	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



$P = \langle a_1, a_4, a_3, a_5 \rangle$

CYCLE

Phase 3: example

$a_1 : \underline{h_4}\ h_5\ h_3\ h_2\ (h_1)$
$a_2 : \underline{h_3}\ h_4\ h_5\ h_9\ h_1\ (h_2)$
$a_3 : (\underline{h_5})\ h_4\ h_1\ h_2\ h_3$
$a_4 : (\underline{h_3})\ h_5\ h_4$
$a_5 : (\underline{h_4})\ h_3\ h_5$
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ (h_6)\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ (h_7)\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ (h_8)$
$a_9 : \underline{h_4}\ h_3\ h_5\ (h_9)$

applicant	counter	house	label
a_1	1	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	

a_1 

$P=\langle a_1 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$
$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$
$a_4 : h_3\ h_5\ h_4$
$a_5 : h_4\ h_3\ h_5$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$
$a_9 : h_4\ h_3\ h_5\ h_9$

applicant	counter	house	label
a_1	1	h_1	
a_2	1	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	

EXTEND

$P=\langle a_1, a_2 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$

$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$

$a_4 : h_3\ h_5\ h_4$

$a_5 : h_4\ h_3\ h_5$

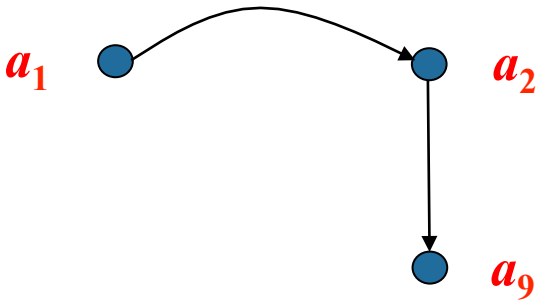
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ h_9$

applicant	counter	house	label
a_1	1	h_1	
a_2	1	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	1	h_9	



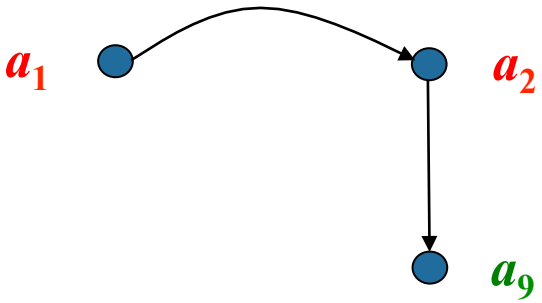
EXTEND

$P = \langle a_1, a_2, a_9 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$
$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$
$a_4 : h_3\ h_5\ h_4$
$a_5 : h_4\ h_3\ h_5$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$
$a_9 : h_4\ h_3\ h_5\ h_9$

applicant	counter	house	label
a_1	1	h_1	
a_2	1	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	1	h_9	



$P=\langle a_1, a_2 \rangle$

BACKTRACK

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$
$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$
$a_4 : h_3\ h_5\ h_4$
$a_5 : h_4\ h_3\ h_5$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$
$a_9 : h_4\ h_3\ h_5\ h_9$

applicant	counter	house	label
a_1	1	h_1	
a_2	1	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	

$P=\langle a_1, a_2 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$

$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$

$a_4 : h_3\ h_5\ h_4$

$a_5 : h_4\ h_3\ h_5$

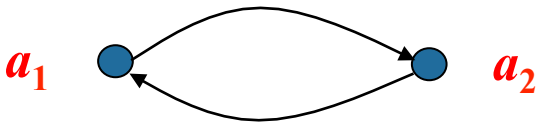
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ h_9$

applicant	counter	house	label
a_1	2	h_1	
a_2	1	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



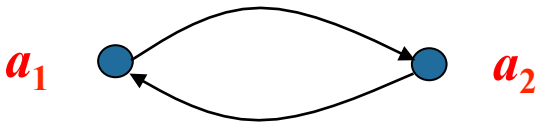
EXTEND

$P = \langle a_1, a_2, a_1 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ h_2\ h_1$
$a_2 : h_3\ h_4\ h_5\ h_9\ h_1\ h_2$
$a_3 : h_5\ h_4\ h_1\ h_2\ h_3$
$a_4 : h_3\ h_5\ h_4$
$a_5 : h_4\ h_3\ h_5$
$a_6 : h_2\ h_3\ h_5\ h_8\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : h_1\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$
$a_9 : h_4\ h_3\ h_5\ h_9$

applicant	counter	house	label
a_1	2	h_1	
a_2	1	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



CYCLE

$P=\langle a_1, a_2 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

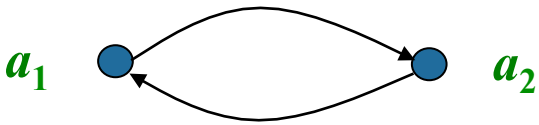
$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ \underline{h_6}\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ \underline{h_7}\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ \underline{h_8}$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



CYCLE

$P = \langle a_1, a_2 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

$a_6 : \underline{h_2}\ h_3\ h_5\ h_8\ \underline{h_6}\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ \underline{h_7}\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ \underline{h_8}$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	1	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	

a_6

$P=\langle a_6 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

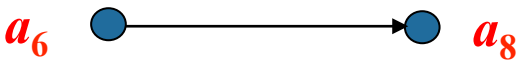
$a_6 : h_2\ h_3\ h_5\ \underline{h_8}\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : \underline{h_1}\ h_5\ h_4\ h_3\ h_7\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	1	h_6	
a_7	0	h_7	
a_8	1	h_8	
a_9	0	h_9	



EXTEND

$P = \langle a_6, a_8 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

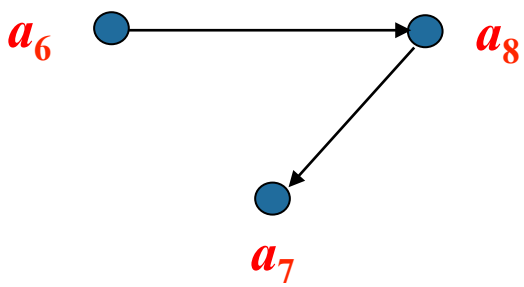
$a_6 : h_2\ h_3\ h_5\ \underline{h_8}\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : \underline{h_1}\ h_4\ h_3\ h_6\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ \underline{h_7}\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	1	h_6	
a_7	1	h_7	
a_8	1	h_8	
a_9	0	h_9	



$P = \langle a_6, a_8, a_7 \rangle$

EXTEND

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

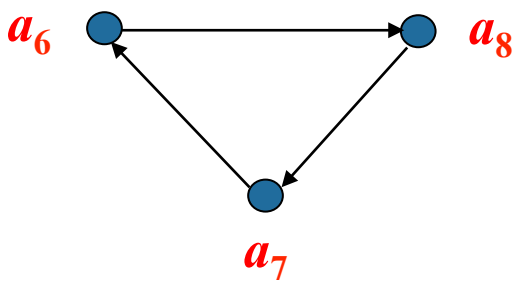
$a_6 : h_2\ h_3\ h_5\ \underline{h_8}\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ \underline{h_6}\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ \underline{h_7}\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	2	h_6	
a_7	1	h_7	
a_8	1	h_8	
a_9	0	h_9	



$P = \langle a_6, a_8, a_7, a_6 \rangle$

EXTEND

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

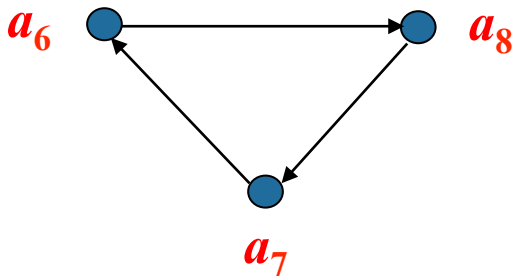
$a_6 : h_2\ h_3\ h_5\ \underline{h_8}\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ \underline{h_6}\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ \underline{h_7}\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	2	h_6	
a_7	1	h_7	
a_8	1	h_8	
a_9	0	h_9	



$P = \langle a_6, a_8, a_7 \rangle$

CYCLE

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

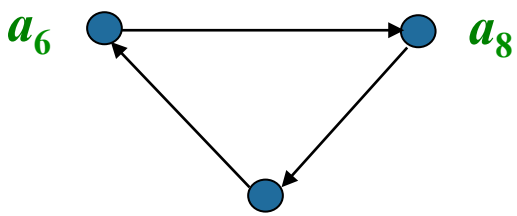
$a_6 : h_2\ h_3\ h_5\ \underline{h_8}\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ \underline{h_6}\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ \underline{h_7}\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	



CYCLE

$P = \langle a_6, a_8, a_7 \rangle$

Phase 3: example

$a_1 : h_4\ h_5\ h_3\ \underline{h_2}\ h_1$

$a_2 : h_3\ h_4\ h_5\ h_9\ \underline{h_1}\ h_2$

$a_3 : \underline{h_5}\ h_4\ h_1\ h_2\ h_3$

$a_4 : \underline{h_3}\ h_5\ h_4$

$a_5 : \underline{h_4}\ h_3\ h_5$

$a_6 : h_2\ h_3\ h_5\ \underline{h_8}\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$

$a_7 : h_1\ h_4\ h_3\ \underline{h_6}\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$

$a_8 : h_1\ h_5\ h_4\ h_3\ \underline{h_7}\ h_6\ h_8$

$a_9 : h_4\ h_3\ h_5\ \underline{h_9}$

applicant	counter	house	label
a_1	0	h_1	
a_2	0	h_2	
a_3	0	h_3	
a_4	0	h_4	
a_5	0	h_5	
a_6	0	h_6	
a_7	0	h_7	
a_8	0	h_8	
a_9	0	h_9	

$P = \langle\rangle$

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Phase 3 termination

- Once Phase 3 terminates, matching is coalition-free
- Data structures may be initialised in $O(m)$ time
- Nested loops take $O(m)$ time overall
- ∴Phase 3 is $O(m)$

$a_1 : h_4\ h_5\ h_3\ (h_2)\ h_1$
$a_2 : h_3\ h_4\ h_5\ h_9\ (h_1)\ h_2$
$a_3 : (h_5)\ h_4\ h_1\ h_2\ h_3$
$a_4 : (h_3)\ h_5\ h_4$
$a_5 : (h_4)\ h_3\ h_5$
$a_6 : h_2\ h_3\ h_5\ (h_8)\ h_6\ h_7\ h_1\ h_{11}\ h_4\ h_{10}$
$a_7 : h_1\ h_4\ h_3\ (h_6)\ h_7\ h_2\ h_{10}\ h_5\ h_{11}$
$a_8 : h_1\ h_5\ h_4\ h_3\ (h_7)\ h_6\ h_8$
$a_9 : h_4\ h_3\ h_5\ (h_9)$

- **Theorem:** given an instance of HA, a maximum Pareto optimal matching can be found in $O(\sqrt{n}m)$ time.

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Housing markets

- Each applicant owns a house initially – their *initial endowment* M_0
- Any matching M must be *individually rational*
  - for each applicant a, either $M(a)=M_0(a)$ or a prefers M to M_0
- An individually rational matching M is in the *strict core* if there is no other matching M' that *weakly blocks* with respect to the *coalition* S
  - the members of S can only improve by exchanging their own resources
  - some member of S prefers M' to M
  - no member of S prefers M to M'
- Every Housing Market admits a unique strict core matching (using the TTC algorithm)
  - [Roth and Postlewaite, 1977]
  - [Shapley and Scarf, 1974]
- The TTC algorithm is strategy-proof
  - [Roth, 1982a]

2.1: Pareto optimal matchings

2.2: Popular matchings

2.3: Profile-based optimal matchings

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Popular matchings

- A matching M is *popular* if there is no matching M' such that more applicants prefer M' to M than prefer M to M'
- Define the relation $\leftarrow$ on matchings by $M \leftarrow M'$ if more applicants prefer M to M' than prefer M' to M
- A popular matching is a minimal element in the relation $\leftarrow$
- A popular matching need not exist, e.g.,

$$a_1 : h_1 \quad h_2 \quad h_3$$
$$a_2 : h_1 \quad h_2 \quad h_3$$
$$a_3 : h_1 \quad h_2 \quad h_3$$

Popular matchings may not exist

- A matching M is *popular* if there is no matching M' such that more applicants prefer M' to M than prefer M to M'
- Define the relation $\leftarrow$ on matchings by $M \leftarrow M'$ if more applicants prefer M to M' than prefer M' to M
- A popular matching is a minimal element in the relation $\leftarrow$
- A popular matching need not exist, e.g.,

$$a_1 : (h_1)\ h_2\ h_3$$
$$a_2 : h_1\ (h_2)\ h_3$$
$$a_3 : h_1\ h_2\ (h_3)$$

- The blue matching shown is unique up to symmetry

Popular matchings may not exist

- A matching M is *popular* if there is no matching M' such that more applicants prefer M' to M than prefer M to M'
- Define the relation $\leftarrow$ on matchings by $M \leftarrow M'$ if more applicants prefer M to M' than prefer M' to M (i.e., M' is *more popular* than M)
- A popular matching is a minimal element in the relation $\leftarrow$
- A popular matching need not exist, e.g.,

$a_1 : h_1\ h_2\ h_3$
$a_2 : h_1\ h_2\ h_3$
$a_3 : h_1\ h_2\ h_3$

- The blue matching shown is unique up to symmetry
- The green matching M' satisfies $M' \leftarrow M$

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Popular matchings can have different sizes

$a_1 : h_1 \quad h_4$

$a_2 : h_2 \quad h_5$

$a_3 : h_3 \quad h_4 \quad h_6$

$a_4 : h_1$

$a_5 : h_2$

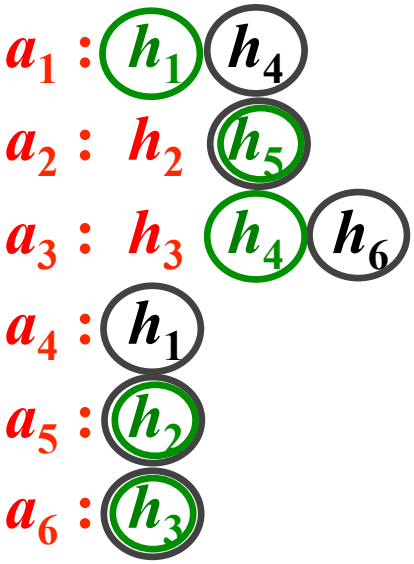
$a_6 : h_3$

Popular matchings can have different sizes

$a_1 : h_1$ (h_4)
$a_2 : h_2$ (h_5)
$a_3 : h_3$ h_4 (h_6)
$a_4 : (h_1)$
$a_5 : (h_2)$
$a_6 : (h_3)$

- The black matching is the unique maximum matching

Popular matchings can have different sizes



- The black matching is the unique maximum matching
- It isn't popular: the green matching M' satisfies $M' \leftarrow M$

Popular matchings can have different sizes

$a_1 : (h_1)\ h_4$
$a_2 :\ h_2\ (h_5)$
$a_3 :\ h_3\ (h_4)\ h_6$
$a_4 :\ h_1$
$a_5 : (h_2)$
$a_6 : (h_3)$

- In fact the green matching is itself popular (size **5**)

Popular matchings can have different sizes

$a_1 : h_1 \; h_4$

$a_2 : h_2 \; h_5$

$a_3 : h_3 \; h_4 \; h_6$

$a_4 : h_1$

$a_5 : h_2$

$a_6 : h_3$

- In fact the green matching is itself popular (size **5**)
- So is the blue matching (size **3**)

Popular matchings can have different sizes

$a_1 : (h_1)\ h_4$
$a_2 :\ h_2\ (h_5)$
$a_3 :\ h_3\ (h_4)\ h_6$
$a_4 :\ h_1$
$a_5 : (h_2)$
$a_6 : (h_3)$

- There is no popular matching of size **6**
- So the green matching is a maximum popular matching

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Summary of results

- Is there a polynomial-time algorithm to determine whether a popular matching exists, and if so to find one?
- Problem introduced by [Gärdenfors, 1975] in the context of two-sided preferences
- Neat (and surprising) characterisation of popular matchings
- Case of strict preferences:
  - $O(n+m)$ algorithm to determine whether a popular matching exists, and if so to find a largest one
- Ties in the preference lists:
  - $O(\sqrt{n}m)$ algorithm for the same problem
- [Abraham, Irving, Kavitha, Mehlhorn, 2005]

Graph representation

- Bipartite graph G, applicant vertices $A = \{a_1, \ldots, a_r\}$, house vertices $H = \{h_1, \ldots, h_s\}$
- Edge $\{a_i, h_j\}$ of weight k if h_j is the k^{th} choice of a_i
- E_k is the set of all edges of weight k
- Example:

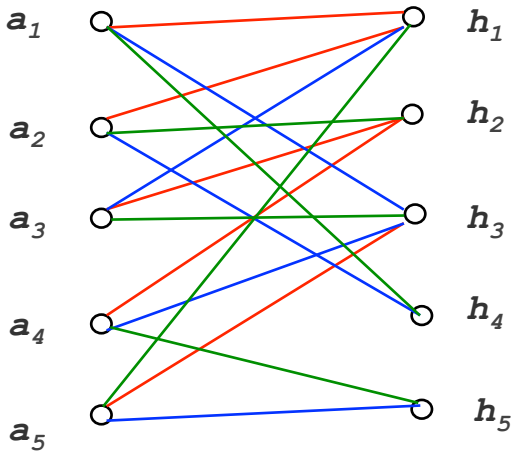
a_1: h_1 h_3 h_4

a_2: h_1 h_4 h_2

a_3: h_2 h_1 h_3

a_4: h_2 h_3 h_5

a_5: h_3 h_5 h_1



E_1 in red

E_2 in blue

E_3 in green

Last resorts

- For each applicant a append a unique house $l(a)$ to the end of his list – his *last resort*
  - This ensures every maximal matching is *applicant-complete*
  - The *size* of a matching is the number of applicants not matched to their last resort

Example:

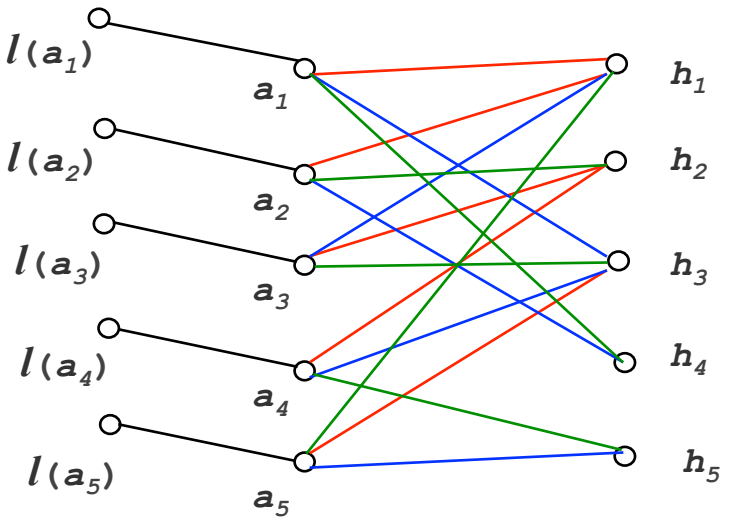
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



E_1 in red

E_2 in blue

E_3 in green

E_4 in black

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f-houses and s-houses

- For applicant a, $f(a)$ denotes his first choice house
- House h is an ***f-house*** if $h = f(a)$ for some a
- For applicant a, $s(a)$ denotes the first non *f*-house on his list
  - $s(a)$ is bound to exist (because of $l(a)$)

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- For applicant a, $f(a)$ denotes his first choice house
- House h is an ***f-house*** if $h = f(a)$ for some a
- For applicant a, $s(a)$ denotes the first non *f*-house on his list
  - $s(a)$ is bound to exist (because of $l(a)$)

Example:

a_1: h_1 h_3 h_4 $l(a_1)$
a_2: h_1 h_4 h_2 $l(a_2)$
a_3: h_2 h_1 h_3 $l(a_3)$
a_4: h_2 h_3 h_5 $l(a_4)$
a_5: h_3 h_5 h_1 $l(a_5)$

- f-houses in green – f-houses are h_1, h_2 and h_3

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f-houses and s-houses

- For applicant a, $f(a)$ denotes his first choice house
- House h is an ***f-house*** if $h = f(a)$ for some a
- For applicant a, $s(a)$ denotes the first non *f*-house on his list
  - $s(a)$ is bound to exist (because of $l(a)$)

Example:

a_1: h_1 h_3 h_4 $l(a_1)$
a_2: h_1 h_4 h_2 $l(a_2)$
a_3: h_2 h_1 h_3 $l(a_3)$
a_4: h_2 h_3 h_5 $l(a_4)$
a_5: h_3 h_5 h_1 $l(a_5)$

- f-houses in green – f-houses are h_1, h_2 and h_3
- s-houses in blue – s-houses are $l(a_3)$, h_4 and h_5

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Lemmas and Theorem

- **Lemma 1**: If M is a popular matching, and h is an f-house then $M(h) = a$ for some a such that $h = f(a)$.

- **Lemma 2**: If M is a popular matching, then $M(a)$ cannot lie between $f(a)$ and $s(a)$ on a's preference list.

- **Lemma 3**: If M is a popular matching, then $M(a)$ is never below $s(a)$ on a's preference list.

- **Theorem 1**: M is a popular matching if and only if

  (i) every f-house is matched in M, and

  (ii) for each applicant a, $M(a) = f(a)$ or $M(a) = s(a)$.

The reduced graph G'

- For each applicant vertex a_i, delete all incident edges except those to $f(a_i)$ and $s(a_i)$

Example

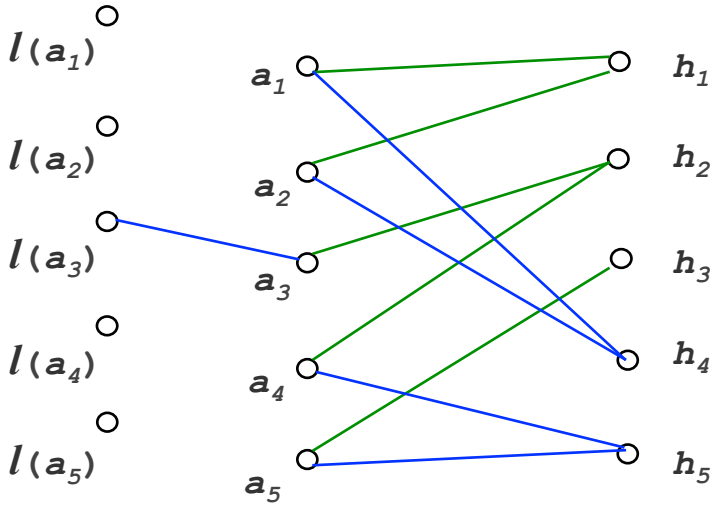
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



$f(a_i)$ in green

$s(a_i)$ in blue

$(a_i, f(a_i))$ in green

$(a_i, s(a_i))$ in blue

The reduced graph G'

- **Theorem 2**: Matching M is popular if and only if

  (i) every f-house is matched in M; and

  (ii) M is an applicant-complete matching in the reduced graph G'

Example

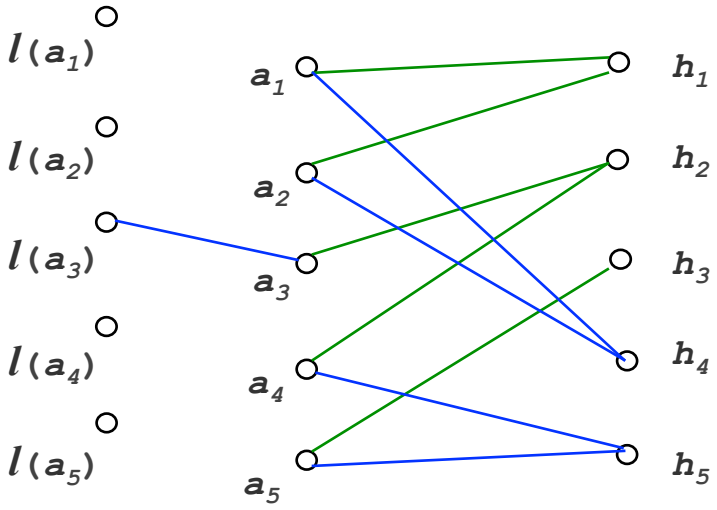
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



The reduced graph G'

- **Theorem 2**: Matching M is popular if and only if

  (i) every f-house is matched in M; and

  (ii) M is an applicant-complete matching in the reduced graph G'

Example

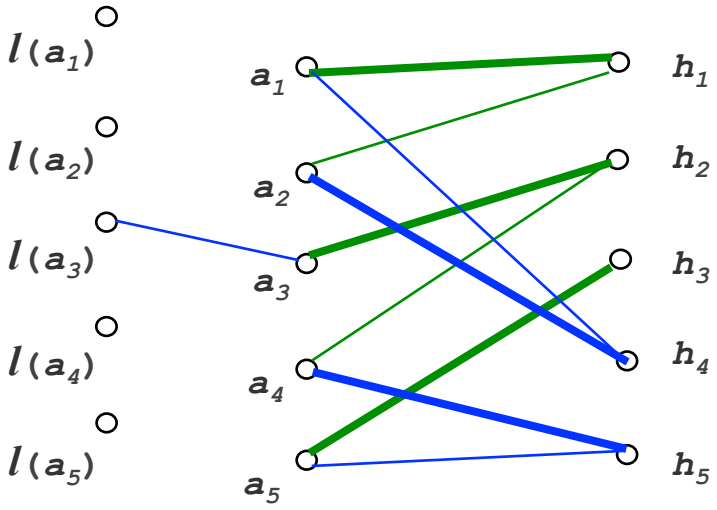
a_1 : h_1 h_3 h_4 $l(a_1)$

a_2 : h_1 h_4 h_2 $l(a_2)$

a_3 : h_2 h_1 h_3 $l(a_3)$

a_4 : h_2 h_3 h_5 $l(a_4)$

a_5 : h_3 h_5 h_1 $l(a_5)$



Popular matching algorithm

```
form G = (V = A ∪ H, E), the graph of the instance;
form G', the reduced graph of G;
if G' admits an applicant-complete matching M   (*)
   for each f-house h unmatched in M
      a = any applicant in f(h);
      promote a to h in M;
else
   no popular matching exists;
```

- Complexity:
  - all steps except (*) are clearly $O(m + n)$
    - $O(m)$ if $n = O(m)$
  - standard matching algorithm for (*) no better than $O(n^{3/2})$

Finding an applicant-complete matching in G'

```
M = ∅;
while some house h has degree 1
   a = unique applicant adjacent to h;
   M = M ∪ {(a,h)};
   G' = G' - {a, h};   // remove a and h from G'
while some house h has degree 0
   G' = G' - {h};
// every surviving house has degree at least 2
// every surviving applicant also has degree 2
if |surviving houses| < |surviving applicants|
   no applicant-complete matching exists;
else
   // G' is a family of disjoint cycles
   return M ∪ any maximum matching of G';
```

- all steps are $O(n + m)$

Example execution

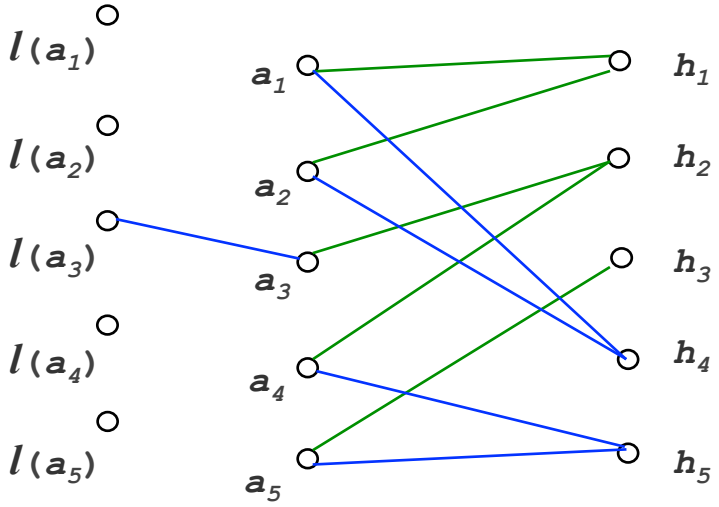
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \emptyset$
  - house $l(a_3)$ has degree 1; add $(a_3, l(a_3))$ to M
  - remove a_3 and $l(a_3)$ from G'

Example execution

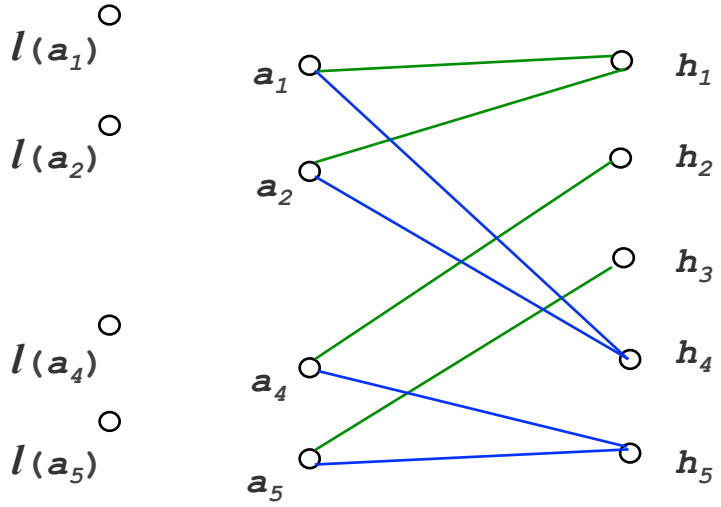
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_3, l(a_3))\}$
  - house h_2 has degree 1; add (a_4, h_2) to M
  - remove a_4 and h_2 from G'

Example execution

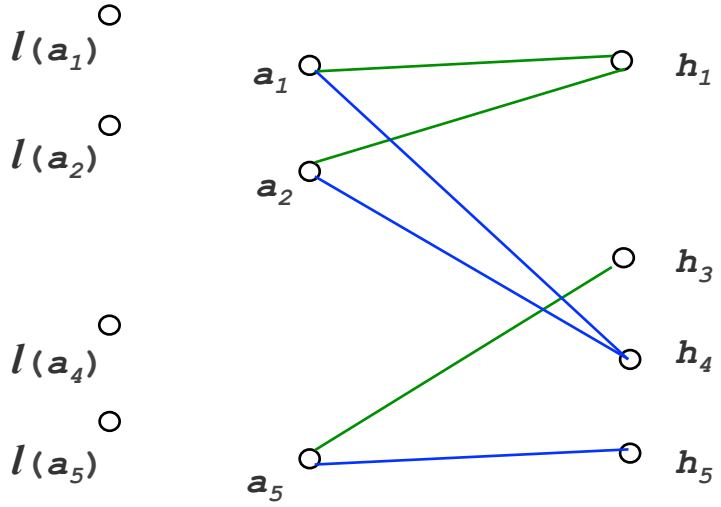
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_3, l(a_3)), (a_4, h_2)\}$
  - house h_5 has degree 1; add (a_5, h_5) to M
  - remove a_5 and h_5 from G'

Example execution

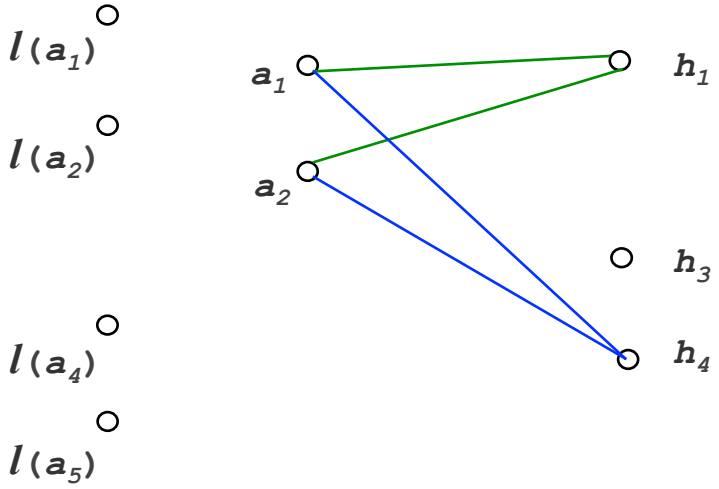
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_3, l(a_3)), (a_4, h_2), (a_5, h_5)\}$
  - remove degree zero houses h_3, $l(a_1)$, $l(a_2)$, $l(a_4)$, $l(a_5)$

Example execution

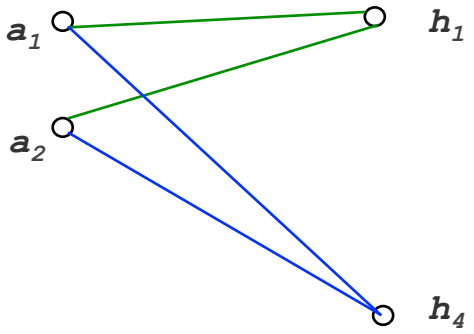
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_3, l(a_3)), (a_4, h_2), (a_5, h_5)\}$
  - no. of surviving houses = no. of surviving applicants
  - G' consists of just one cycle;
  - traverse the cycle adding alternate edges to M;

Example execution

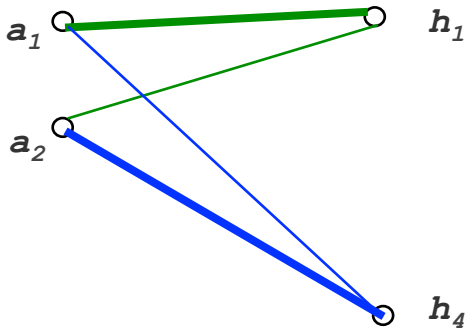
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_1, h_1), (a_2, h_4), (a_3, l(a_3)), (a_4, h_2), (a_5, h_5)\}$

Example execution

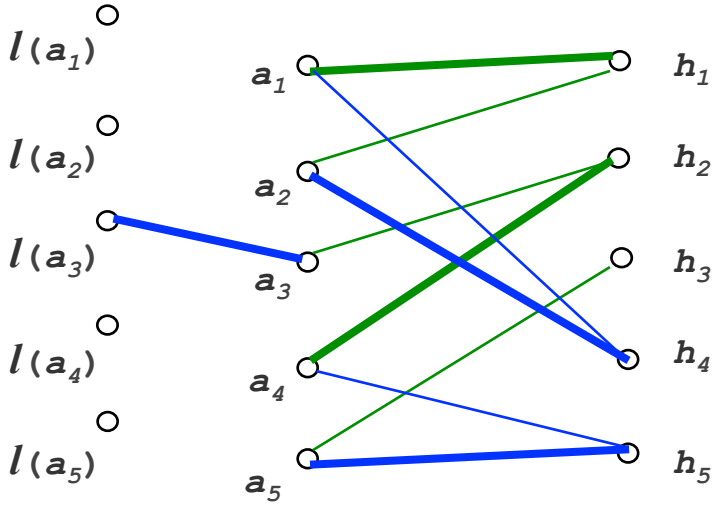
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_1, h_1), (a_2, h_4), (a_3, l(a_3)), (a_4, h_2), (a_5, h_5)\}$
  - h_3 is an unmatched f-house in M;
  - promote a_5 to h_3

Example execution

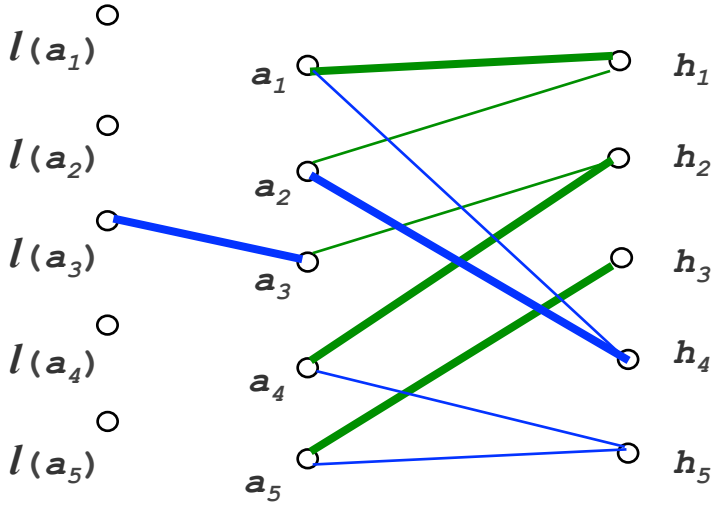
a_1: h_1 h_3 h_4 $l(a_1)$

a_2: h_1 h_4 h_2 $l(a_2)$

a_3: h_2 h_1 h_3 $l(a_3)$

a_4: h_2 h_3 h_5 $l(a_4)$

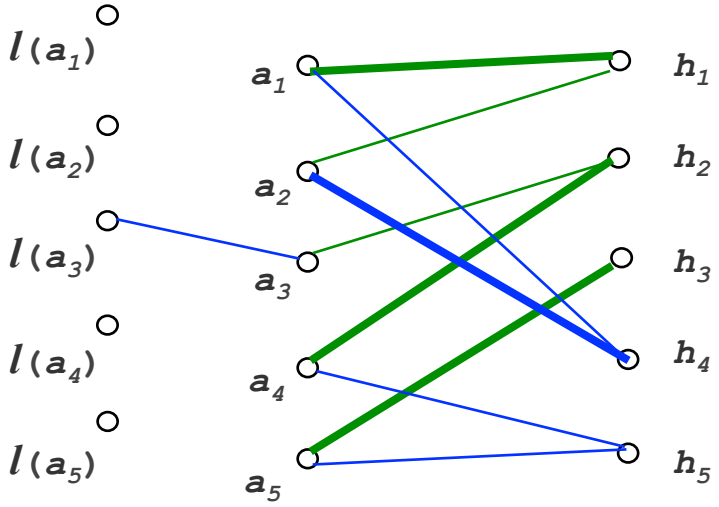
a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_1, h_1), (a_2, h_4), (a_3, l(a_3)), (a_4, h_2), (a_5, h_3)\}$
- Matching M with last resort edges removed is now popular

Example execution

a_1: h_1 h_3 h_4 $l(a_1)$
a_2: h_1 h_4 h_2 $l(a_2)$
a_3: h_2 h_1 h_3 $l(a_3)$
a_4: h_2 h_3 h_5 $l(a_4)$
a_5: h_3 h_5 h_1 $l(a_5)$



- $M = \{(a_1, h_1), (a_2, h_4), (a_4, h_2), (a_5, h_3)\}$
- Matching M is now popular
- **Theorem:** There is an $O(n+m)$ algorithm that finds a popular matching or reports that none exists, given an instance of HA.

- Straightforward extension to find a maximum popular matching
- Non-trivial extension to the case of ties in the preference lists ($O(\sqrt{n}m)$ time)
  - [Abraham, Irving, Kavitha and Mehlhorn, 2005]

Tutorial Outline

2.1: Pareto optimal matchings

2.2: Popular matchings

2.3: Profile-based optimal matchings

The profile of a matching

- The *degree* of a matching M is the maximum k such that some applicant has their kth-choice house in M
- Example:

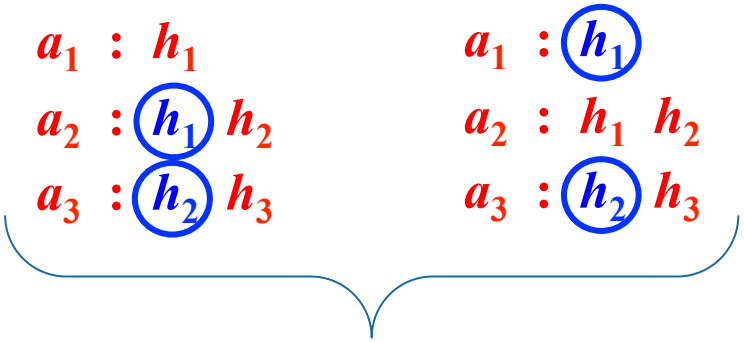
a_1:	**(h_1)**	h_3	h_4
a_2:	h_1	**(h_4)**	h_2
a_3:	h_2	h_1	**(h_3)**
a_4:	**(h_2)**	h_3	h_5
a_5:	h_3	**(h_5)**	h_1

  - $M = \{(a_1, h_1), (a_2, h_4), (a_3, h_3), (a_4, h_2), (a_5, h_5)\}$
  - Degree of M is 3
- The *profile* of M is a vector $\langle x_1, x_2, \ldots, x_k \rangle$ where k is the degree of M and x_i is the number of applicants who have their ith-choice house in M
  - Profile of M is $\langle 2, 2, 1 \rangle$

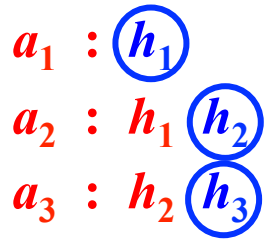
Rank-maximal matchings

- A matching M is *rank-maximal* if its profile is lexicographically maximum
  - So in M, the maximum number of applicants obtain their first-choice house, and subject to this, the maximum number obtain their second-choice house, etc.
  - A rank-maximal matching M can be computed in $O(\min\{r^*\sqrt{n},\ n+r^*\}m)$ time (where $n=|V|$, $m=|E|$, and r^* is the degree of M)
  - [Irving, Kavitha, Mehlhorn, Michail and Paluch, 2004]
- A rank-maximal matching need not be maximum!
- Example

$a_1 : h_1$
$a_2 : h_1\ h_2$
$a_3 : h_2\ h_3$



rank-maximal matchings – profile $\langle 2,0\rangle$



maximum matching – profile $\langle 1,2\rangle$

Other profile-based optimal matchings

- A matching M is *greedy maximum* [Irving, 2007] if
  1. M is maximum
  2. subject to 1, M has lexicographically maximum profile
- A matching M is *generous* [Irving, 2007] if
  1. M is maximum
  2. subject to 1, the reverse profile of M is lexicographically minimum

$a_1 : (h_1)\ h_2\ h_3\ h_4\ h_5$
$a_2 : h_1\ (h_2)\ h_3\ h_4\ h_5$
$a_3 : h_1\ h_2\ h_3\ (h_4)\ h_5$
$a_4 : h_1\ (h_3)\ h_5\ h_4\ h_2$
$a_5 : h_2\ (h_5)\ h_4\ h_3\ h_1$

- Profile $\langle 1,3,0,1 \rangle$
- Weight **11**
- Minimum weight maximum matching

$a_1 : (h_1)\ h_2\ h_3\ h_4\ h_5$
$a_2 : h_1\ h_2\ h_3\ (h_4)\ h_5$
$a_3 : h_1\ h_2\ h_3\ h_4\ (h_5)$
$a_4 : h_1\ (h_3)\ h_5\ h_4\ h_2$
$a_5 : (h_2)\ h_5\ h_4\ h_3\ h_1$

- Profile $\langle 2,1,0,1,1 \rangle$
- Weight **13**
- Greedy maximum matching

$a_1 : (h_1)\ h_2\ h_3\ h_4\ h_5$
$a_2 : h_1\ (h_2)\ h_3\ h_4\ h_5$
$a_3 : h_1\ h_2\ (h_3)\ h_4\ h_5$
$a_4 : h_1\ h_3\ (h_5)\ h_4\ h_2$
$a_5 : h_2\ h_5\ (h_4)\ h_3\ h_1$

- Profile $\langle 1,1,3 \rangle$
- Weight **12**
- Generous matching

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Other profile-based optimal matchings

- A matching M is *greedy maximum* [Irving, 2007] if
  1. M is maximum
  2. subject to 1, M has lexicographically maximum profile
- A matching M is *generous* [Irving, 2007] if
  1. M is maximum
  2. subject to 1, the reverse profile of M is lexicographically minimum
- A matching M is *greedy generous* [Irving, 2007] if
  1. M is maximum
  2. the degree of M equals that of a generous matching
  3. subject to 1 and 2, M has lexicographically maximum profile
- A greedy maximum / generous / greedy generous matching M can be found in $O(r^* \sqrt{n} m \log n)$ time (where r^* is the degree of M, $n=|V|$ and $m=|E|$)
  - [Huang and Kavitha, 2012]

Additional constraints

- Reviewer assignment problem considered by [Garg et al, 2010]
- *Coverage*: there will be a value of $t \geq 1$ such each paper must be reviewed t times
- *Load balancing*: each reviewer should be given roughly the same number of papers
- *Fairness*:

$r_1 : (p_1\ p_2)\ (p_3\ p_4)$

$r_2 : (p_1\ p_2)\ (p_3\ p_4)$

Additional constraints

- Reviewer assignment problem considered by [Garg et al, 2010]
- *Coverage*: there will be a value of $t \geq 1$ such each paper must be reviewed t times
- *Load balancing*: each reviewer should be given roughly the same number of papers
- *Fairness*:

  Both matchings have profile $\langle 2,2 \rangle$ but clearly second is fairer

- Solution: introduce profile for each reviewer
  - $p(r_1,M_1)=\langle 2,0 \rangle$, $p(r_2,M_1)=\langle 0,2 \rangle$
  - $p(r_1,M_2)=\langle 1,1 \rangle$, $p(r_2,M_2)=\langle 1,1 \rangle$
- Create sorted vector of profiles (worst to best according to lexicographic order)
  - $\langle \langle 0,2 \rangle, \langle 2,0 \rangle \rangle$ for M_1
  - $\langle \langle 1,1 \rangle, \langle 1,1 \rangle \rangle$ for M_2

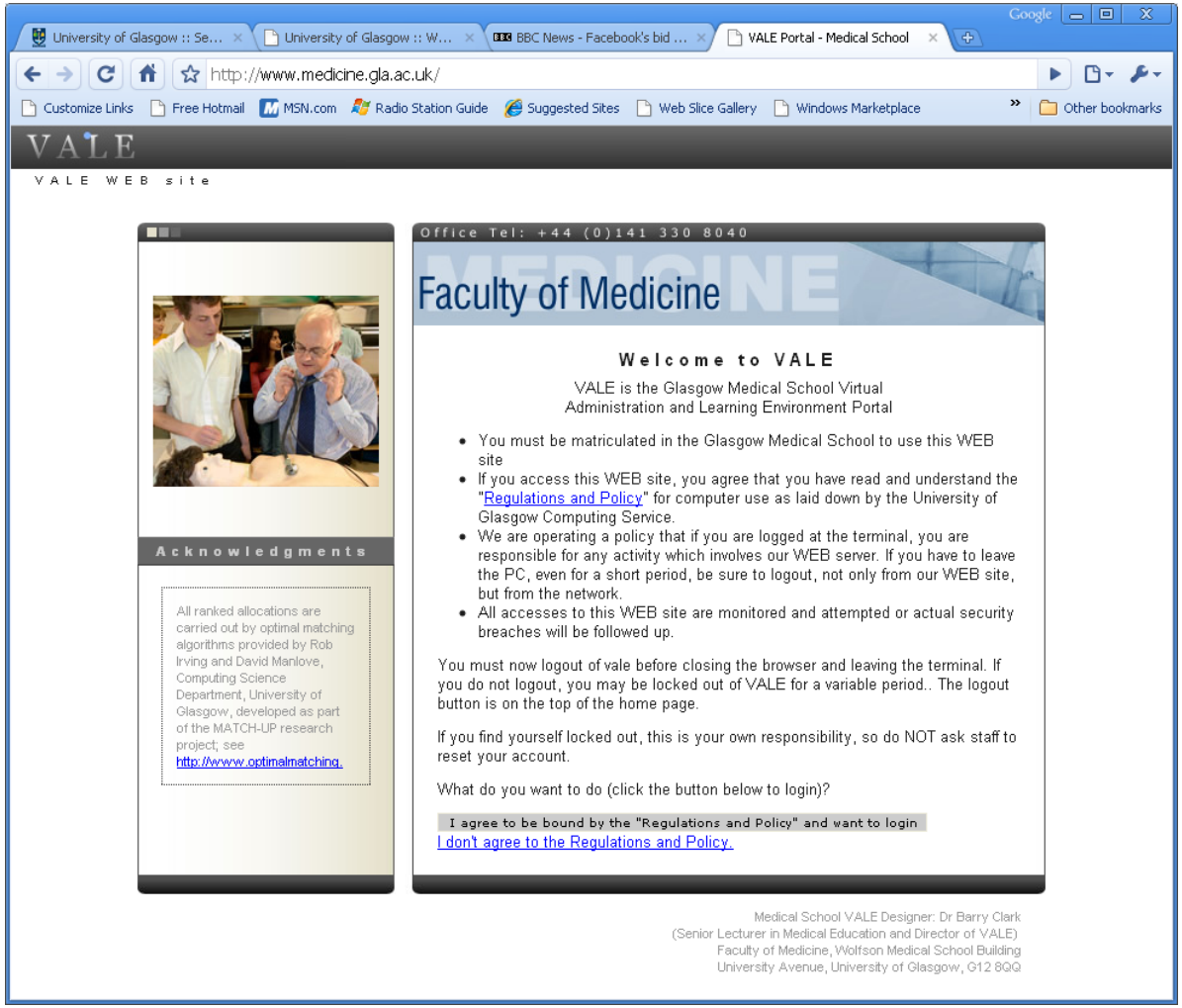
Leximin optimal matchings

- [Garg et al, 2010]: *leximin optimal matching*
  - lexicographic ordering on vectors of sorted profiles
  - $\langle\langle 1,1\rangle, \langle 1,1\rangle\rangle$ is better than $\langle\langle 0,2\rangle, \langle 2,0\rangle\rangle$
  - so M_2 is the leximin optimal matching

- Finding a leximin optimal matching is NP-hard in general, but solvable in polynomial time if maximum rank of a paper is **2** [Garg et al, 2010]

Course allocation at the School of Medicine

- VALE: Virtual Administration and Learning Environment
  - Handles various allocation tasks, including:
    - Allocating medical students to "SSCs" (elective courses)
    - Allocating medical students to short-term hospital placements



- **http://www.medicine.gla.ac.uk**

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2006-07 data

- 246 medical students bidding for 17 SSCs, each with a fixed capacity

SSC #	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Capacity	60	20	30	3	60	4	2	24	4	8	30	3	2	20	10	14	14

- Each preference list of length 6
- Outcome:

	Minimum cost	Greedy	Generous	Greedy generous
Size	246	246	246	246
Weight	463	521	479	478
Profile	⟨119,74,30,13,6,4⟩	⟨151,28,15,13,17,22⟩	⟨95,87,46,18⟩	⟨118,60,32,36⟩

- Serial dictatorship mechanism produced matching with size 228, weight 478 and profile ⟨117,49,21,16,14,11⟩

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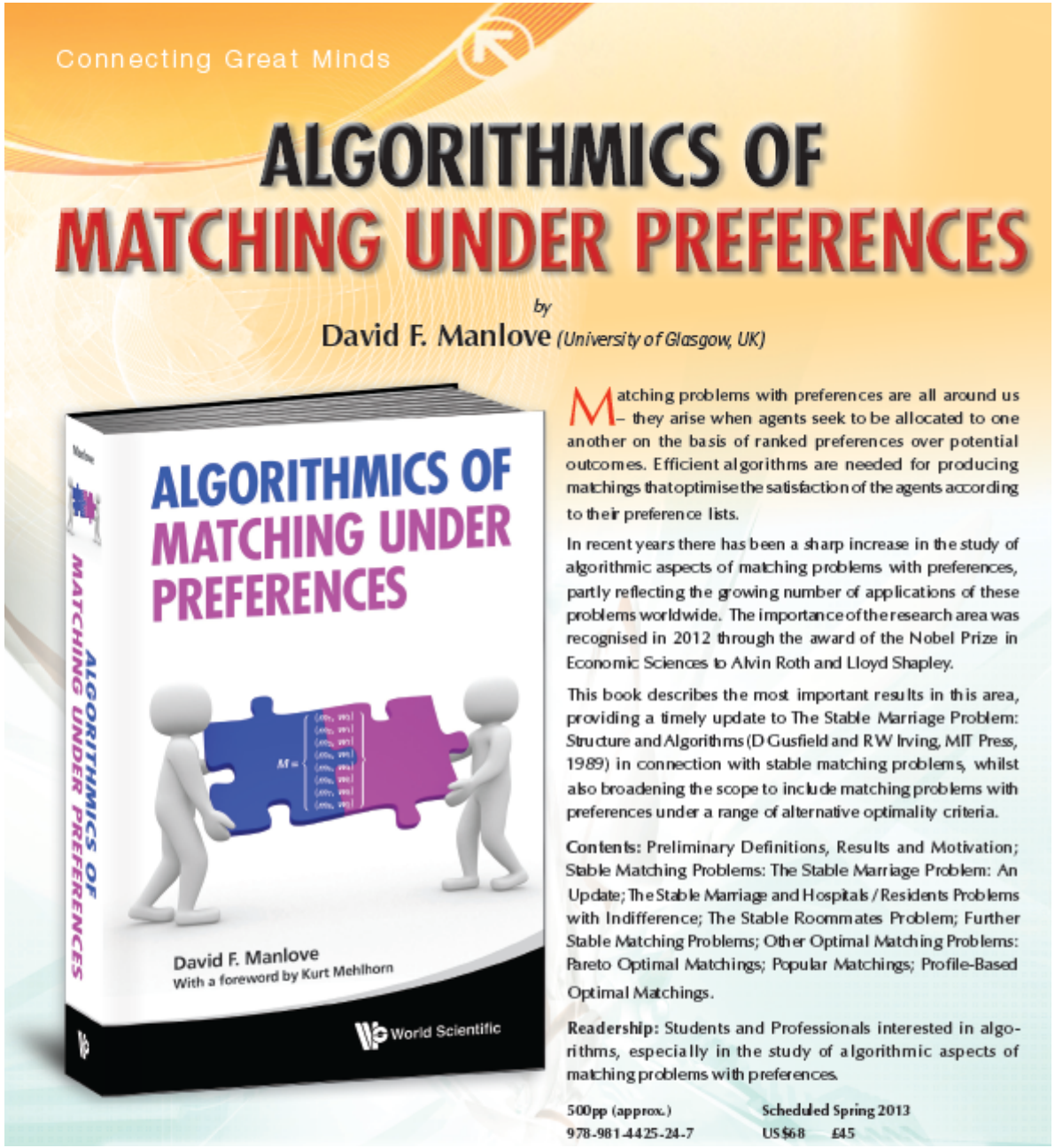
Open problems

- Maximum Pareto optimal matching
  - Ties in the preference lists
    - Solvable in $O(\sqrt{n}m \log n)$ time. Can we achieve $O(\sqrt{n}m)$ time?
- Popular matchings in the Roommates problem
  - Single set of participants
  - Each participant has a (strict) preference list containing a subset of the others
  - Efficient algorithm to find a popular matching or report that none exists?
- Profile-based optimal matchings
  - Structure of the set of rank-maximal matchings
  - Similarly for other types of profile-based optimal matchings
  - Profile-based optimal matchings in the non-bipartite case
- Acknowledgements: Rob Irving; Baharak Rastegari

Further reading



- Chapters 6, 7, 8

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