# The assignment game: core, competitive equilibria and multiple partnership

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# Outline

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    - Dual solutions and the core
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# Coalitional TU games

A coalitional game with transferable utility is $(N, v)$, where

- $N = \{1, 2, \ldots, n\}$ is the set of players and
- $\begin{array}{rcl} v : 2^N & \longrightarrow & \mathbb{R} \\ S & \mapsto & v(S) \end{array}$ is the characteristic function.

An imputation is a payoff vector $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^N$ that is

- Efficient: $\sum_{i \in N} x_i = v(N)$
- Individually rational: $x_i \geq v(i)$ for all $i \in N$.

Let $I(v)$ be the set of imputations of $(N, v)$ and $I^*(v)$ be the set of preimputations (efficient payoff vectors).

# The core

Let it be $(N, v)$ and $x, y \in I^*(v)$:

- $y$ dominates $x$ via coalition $S \neq \emptyset$ $(y\, dom_S^v x) \Leftrightarrow x_i < y_i$ for all $i \in S$ and $\sum_{i \in S} y_i \leq v(S)$.
- $y$ dominates $x$ $(y\, dom^v x)$ if $y\, dom_S^v x$ for some $S \subseteq N$.

**Definition (Gillies, 1959)**

The core $C(v)$ of $(N, v)$ is the set of preimputations undominated by another preimputation.

- If $C(v) \neq \emptyset$, then it coincides with the set of imputations undominated by another imputation.
- Equivalently, $C(v) = \{x \in I(v) \mid \sum_{i \in S} x_i \geq v(S), \text{ for all } S \subseteq N\}$.

# The assignment game (Shapley and Shubik, 1972)

- The assignment game is a cooperative model for a two-sided market (Shapley and Shubik, 1972).
- A good is traded in indivisible units.
- Side payments are allowed and utility is identified with money.
- Each buyer in $M = \{1, 2, \dots, m\}$ demands one unit and each seller in $M' = \{1, 2, \dots, m'\}$ supplies one unit.
- Each seller $j \in M'$ has a reservation value $c_j \geq 0$ for his object.
- Each buyer $i \in M$ valuates differently, $h_{ij} \geq 0$, the object of each seller $j$.
- Buyer $i$ and seller $j$, whenever they trade, make a join profit of $(h_{ij} - p) + (p - c_j)$. Hence, $a_{ij} = \max\{0, h_{ij} - c_j\}$.

All these data is summarized in the assignment matrix $A$:

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m'} \\ a_{21} & a_{22} & \dots & a_{2m'} \\ \dots & \dots & \dots & \dots \\ a_{m1} & a_{m2} & \dots & a_{mm'} \end{pmatrix}$$

# The assignment game

Cooperation means we look at this market as a centralized market where a matching of buyers to sellers and a distribution of the profit of this matching is proposed: $(u, v) \in \mathbb{R}^{M} \times \mathbb{R}^{M'}$.

✓ A matching $\mu$ is a subset of $M \times M'$ where each agent appears in at most one pair. Let $\mathcal{M}(M, M')$ be the set of matchings.

✓ A matching $\mu$ is optimal iff, for any other $\mu' \in \mathcal{M}(M, M')$,

$$\sum_{(i,j)\in\mu} a_{ij} \geq \sum_{(i,j)\in\mu'} a_{ij}.$$

Let $\mathcal{M}^{*}_{A}(M, M')$ be the set of optimal matchings.

The cooperative assignment game is defined by $(M \cup M', w_A)$, the characteristic function $w_A$ being (for all $S \subseteq M$ and $T \subseteq M'$)

$$w_A(S \cup T) = \max\{ \sum_{(i,j)\in\mu} a_{ij} \mid \mu \in \mathcal{M}(S, T)\}.$$

# The core

The core:

$$C(w_A) = \left\{ (u,v) \in \mathbb{R}^M \times \mathbb{R}^{M'} \;\middle|\; \begin{array}{l} \sum_{i\in M} u_i + \sum_{j\in M'} v_j = w_A(M \cup M') \\ u_i + v_j \geq a_{ij} \text{ for all } (i,j) \in M \times M', \\ u_i \geq 0, \forall i \in M, v_j \geq 0, \forall j \in M'. \end{array} \right\}$$

Given any optimal matching $\mu$, if $(u,v) \in C(w_A)$ then $u_i + v_j = a_{ij}$ for all $(i,j) \in \mu$ and $u_i = 0$ if $i$ is unmatched by $\mu$.

**Fact**

*In the core of the assignment game, third-party payments are excluded*

# The core

**Theorem (Shapley and Shubik, 1972)**

*The core of the assignment game is non-empty and coincides with the set of solutions of the dual program to the linear assignment problem.*

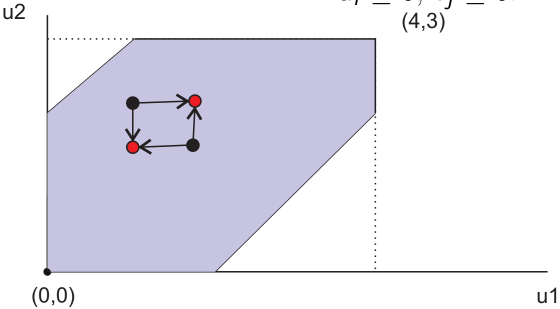
$$\begin{array}{lll}
w_A(M \cup M') & = \max \sum_{i \in M} \sum_{j \in M'} a_{ij} x_{ij} & \min \sum_{i \in M} u_i + \sum_{j \in M'} v_j \\
\text{where} & \sum_{i \in M} x_{ij} \leq 1,\ \forall\, j \in M', & u_i + v_j \geq a_{ij}\ \forall (i,j) \in M \times M', \\
 & \sum_{j \in M'} x_{ij} \leq 1,\ \forall\, i \in M, & u_i \geq 0,\ v_j \geq 0\,. \\
 & x_{ij} \geq 0,\ \forall\, (i,j) \in M \times M'. &
\end{array}$$

# Example 1

|   | 3 | 4 |
|---|---|---|
| 1 | **4** | 1 |
| 2 | 2 | **3** |

$$\begin{aligned} u_1 + v_3 &= 4 \\ u_1 + v_4 &\geq 1 \\ u_2 + v_3 &\geq 2 \\ u_2 + v_4 &= 3 \\ u_i \geq 0,\ v_j &\geq 0. \end{aligned}$$

$$\begin{aligned} -2 \leq u_2 - u_1 &\leq 2 \\ 0 \leq u_1 &\leq 4 \\ 0 \leq u_2 &\leq 3 \end{aligned}$$



- $(\overline{u}, \underline{v})$ and $(\underline{u}, \overline{v})$, optimal core points for each side.
- $(\overline{u}, \underline{v}) = (4, 3; 0, 0)$, $(\underline{u}, \overline{v}) = (0, 0, 4, 3)$.

# Lattice structure 1

**Fact (Shapley and Shubik, 1972)**

$C(w_A)$ *with the following partial order(s) is a complete lattice*

$$(u,v) \leq_M (u',v') \Leftrightarrow u_i \leq u'_i \quad \forall i \in M.$$

Let $(M \cup M', w_A)$ be an assignment market and $(u,v)$, $(u',v')$ two elements in $C(w_A)$. Then,

$$(u,v) \vee (u',v') = \left( (\max\{u_i, u'_i\})_{i\in M}, (\min\{v_j, v'_j\})_{j\in M'} \right) \in C(w_A)$$
$$(u,v) \wedge (u',v') = \left( (\min\{u_i, u'_i\})_{i\in M}, (\max\{v_j, v'_j\})_{j\in M'} \right) \in C(w_A).$$

✓ As a consequence the existence of a buyers-optimal core allocation and a sellers-optimal core allocation is obtained.

**Fact (Demange, 1982; Leonard, 1983)**

*For all* $i \in M$, $\overline{u}_i = w_A(M \cup M') - w_A(M \cup M' \setminus \{i\})$.

# The buyers-optimal core allocation

- The buyers optimal core allocation $(\overline{u}, \underline{v})$ can be obtained by solving $m+1$ linear programs.
- But since all buyers attain their marginal contribution at the same core point, it can easily be obtained by means of only two linear programs: the one that gives an optimal matching $\mu$ and

$$
\begin{array}{ll}
\max & \sum_{i \in M} u_i \\
\text{where} & u_i + v_j \geq a_{ij} \, \forall (i,j) \in M \times M', \\
 & u_i + v_j = a_{ij}, \forall (i,j) \in \mu, \\
 & u_i \geq 0, \; v_j \geq 0 \,.
\end{array}
$$

# Competitive equilibria

✓ In this section let us interpret $M$ as a set of bidders and $M'$ as a set of objects.

✓ A feasible price vector is $p \in \mathbb{R}^{M'}$ such that $p_j \geq c_j$ for all $j \in M'$.

✓ Add a null object $O$ with $a_{iO} = 0$ for all $i \in M$ and price 0. More than one bidder may be matched to $O$: $Q = M' \cup \{O\}$.

✓ The demand set of a bidder $i$ at prices $p$ is

$$D_i(p) = \left\{ j \in Q \mid a_{ij} - p_j = \max_{k \in Q} \{a_{ik} - p_k\} \right\}.$$

✓ The price vector $p$ is quasi-competitive if there is a matching $\mu$ such that, for all $i \in M$, if $\mu(i) = j$ then $j \in D_i(p)$. Then $\mu$ is compatible with $p$.

✓ $(p, \mu)$ is a competitive equilibrium if $p$ is a quasi-competitive price, $\mu$ is compatible with $p$ and $p_j = c_j$ for all $j \notin \mu(M)$.

# Competitive equilibria

**Theorem (Gale, 1960)**

*Let $(M, M', A)$ be an assignment market. Then,*

1. $(p, \mu)$ *competitive equilibrium* $\Rightarrow$ $(u, v) \in C(w_A)$ *where* $u_i = h_{ij} - p_j$ *if* $\mu(i) = j$; $v_j = p_j - c_j, j \in M' \setminus \{O\}$
2. $\mu \in \mathcal{M}^*_A(M, Q)$ *with* $a_{i\mu(i)} > 0 \, \forall i \in M$ *and* $(u, v) \in C(w_A)$ $\Rightarrow$ $(p, \mu)$ *is a competitive equilibrium, where* $p_j = v_j + c_j$ *if* $j \in M'$ *and* $p_O = 0$

✓ The buyers-optimal core allocation corresponds to the minimal competitive price vector.

✓ The sellers-optimal core allocation corresponds to the maximal competitive price vector.

# Lattice structure 2

Given a (square) assignment market $(M, M', A)$, denote by $i'$ the $i$th seller and assume $\mu = \{(i, i') \mid i \in M\}$ is optimal. Then, the projection of $C(w_A)$ to the space of the buyers' payoffs is

$$C_u(w_A) = \left\{ u \in \mathbb{R}^M \;\middle|\; \begin{array}{l} a_{ij} - a_{jj} \leq u_i - u_j \leq a_{ii} - a_{ji} \, \forall i, j \in \{1, 2, \ldots, m\} \\ 0 \leq u_i \leq a_{ii} \text{ for all } i \in \{1, 2, \ldots, m\}. \end{array} \right\}$$

✓ Notice that $C_u(w_A)$ is a 45-degree lattice.

**Theorem (Quint, 1991; Characterization of the core )**

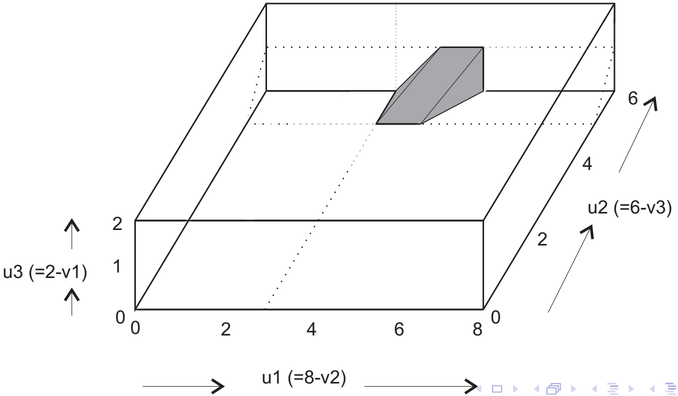
*Given any 45-degree lattice $L$, there exists an assignment game $(M, M', A)$ such that $C(w_A) = L$.*

✓ But matrix $A$ in the above theorem may not be unique.

# Example 2

| | 1' | 2' | 3' |
|---|---|---|---|
| 1 | 5 | **8** | 2 |
| 2 | 7 | 9 | **6** |
| 3 | **2** | 3 | 0 |

- ✓ Optimal matching: $\mu = \{(1,2'),(2,3'),(3,1')\}$.
- ✓ $(\overline{u},\underline{v}) = (5,6,1;1,3,0)$, $(\underline{u},\overline{v}) = (3,5,0;2,5,1)$.



# Example 2

$A^{\alpha}$:

$$\begin{array}{c|c|c|c|}
 & 1' & 2' & 3' \\ \hline
1 & 5 & \mathbf{8} & \alpha \\ \hline
2 & 7 & 9 & \mathbf{6} \\ \hline
3 & \mathbf{2} & 3 & 0 \\ \hline
\end{array}$$

✓ Notice that for all $(u, v) \in C(w_A)$, $u_1 + v_3 \geq 3 > 2$:

$$u_1 + v_3 = u_1 + v_1 + u_3 + v_3 - u_3 - v_1 \geq a_{11} + a_{33} - a_{31} = 5 + 0 - 2 = 3.$$

✓ Hence, all matrices $A^{\alpha}$ with $\alpha \in [0, 3]$ lead to assignment markets with the same core.

# Some properties of the core

**Definition (Solymosi and Raghavan, 2001)**

$(M, M', A)$ a square assignment market and $\mu \in \mathcal{M}^*_A(M, M')$:

1. $A$ has dominant diagonal $\Leftrightarrow$ $a_{i\mu(i)} \geq \max\{a_{ij}, a_{k,\mu(i)}\}$ for all $i, k \in M$, $j \in M'$.
2. $A$ has a doubly dominant diagonal $\Leftrightarrow$ $a_{ij} + a_{k\mu(k)} \geq a_{i\mu(k)} + a_{kj}$ for all $i, k \in M$ and $j \in M'$.

**Theorem (Solymosi and Raghavan, 2001)**

*Let $(M, M', A)$ be a square assignment market. $C(w_A)$ is stable ($\forall x \in I(w_A) \setminus C(w_A)$, $\exists y \in C(w_A)$, $y \, dom x$) $\Leftrightarrow$ $A$ has a dominant diagonal.*

# Markets with the same core

**Definition**

An assignment market $(M, M', A)$ is buyer-seller exact $\Leftrightarrow$ for all $(i, j) \in M \times M'$ there exists $(u, v) \in C(w_A)$ such that $u_i + v_j = a_{ij}$.

**Fact (Núñez and Rafels, 2002)**

*An assignment market* $(M, M', A)$ *is buyer-seller exact* $\Leftrightarrow$ $A$ *has a doubly dominant diagonal.*

**Fact (Martínez-de-Albéniz, Núñez and Rafels, 2011)**

*Two square assignment markets* $(M, M', A)$ *and* $(M, M', B)$ *have the same core* $\Leftrightarrow$ *for all* $(i, j) \in M \times M'$

$$w_A(N \setminus \{i, j\}) = w_B(N \setminus \{i, j\}).$$

# Markets with the same core

**Theorem (Martínez-de-Albéniz, Núñez and Rafels, 2011)**

*The set of matrices leading to markets with the same core as* $(M, M', A)$ *is a join-semilattice* $(\langle A \rangle, \leq)$ *with one maximal element an a finite number of minimal elements:*

$$\langle A \rangle = \bigcup_{q=1}^{p} [A_q, \overline{A}].$$

In Example 2:

$$\langle A \rangle = \left[ \begin{pmatrix} 5 & 8 & 0 \\ 7 & 9 & 6 \\ 2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 5 & 8 & 3 \\ 7 & 9 & 6 \\ 2 & 3 & 0 \end{pmatrix} \right]$$

# More References

1. On the extreme core points:
   - Balinsky and Gale (1987).
   - Hamers et al. (2002) prove that every extreme core allocation is a marginal worth vector.
   - Characterization as the set of reduced marginal worth vectors (Núñez and Rafels, 2003).
   - A computation procedure (Izquierdo, Núñez and Rafels, 2007).
2. On the dimension of the core: Núñez and Rafels, 2008.
3. Axiomatic characterizations of the core (on the class of assignment games with reservation values; Owen, 1992):
   - There is a first axiomatization of the core due to Sasaki (1995).
   - Toda (2003): *Pareto optimality, individual rationality, (derived) consistency and super-additivity*.
   - Toda (2005): *Pareto optimality, (projected) consistency, pairwise monotonicity and individual monotonicity (or population monotonicity)*.
   - The core is the only solution satisfying *derived consistency and Toda's consistency* (Llerena, Núñez and Rafels, 2013).

# Multiple-partners assignment market: Model 1 (Sotomayor, 1992)

A multiple partner assignment game is $M_1(F_0, W_0, \alpha, r, s)$ where

- $F$ is the finite set of firms and $W$ the finite set of workers.
- Firm $i$ hires at most $r_i$ workers and worker $k$ has at most $s_k$ jobs.
- $\alpha_{ik} \geq 0$ the income the pair $(i, k)$ generates if they work together.
- If firm $i$ hires worker $k$ at a salary $v_{ik}$, its profit is $u_{ik} = \alpha_{ik} - v_{ik}$.
- As many copies of a dummy firm $f_0$ and a dummy worker $w_0$ as needed. $F_0$ and $W_0$ are the sets of firms and workers with the respective dummy agents.

# $M_1$: Outcomes

**Definition**

A feasible matching $x$ is a $m \times n$ matrix $(x_{ik})_{(i,k)\in F\times W}$ with $x_{ik} \in \{0,1\}$ such that

- $\sum_{k\in W} x_{ik} \leq r_i$ for all $i \in F$,
- $\sum_{i\in F} x_{ik} \leq s_k$ for all $k \in W$, where $x_{ik} = 1$ means that $i$ and $k$ form a partnership.

• $C(i,x)$ is the set of workers hired by $i$ under $x$ and as many copies of $w_0$ as necessary ($|C(i,x)| = r_i$).
• If $C(i,x) \cap W = \emptyset$ then $i$ is unmatched by $x$ (or matched only to $w_0$).
An outcome in this market is determined by specifying a matching and the way in which the income within each partnership is divided among its members.

# $M_1$: Pairwise-stability

## Definition

A feasible outcome $((u, v); x)$ is a feasible matching $x$ and a set of numbers $u_{ik}$ and $v_{ik}$, for $(i, k) \in F_0 \times W_0$ with $x_{ik} = 1$, such that

- $u_{ik} + v_{ik} = \alpha_{ik}$, $u_{ik} \geq 0$, $v_{ik} \geq 0$ for all $(i, k) \in F \times W$ with $x_{ik} = 1$.
- $u_{iw_0} = u_{f_0 k} = u_{f_0 w_0} = 0$, $v_{f_0 k} = v_{iw_0} = v_{f_0 w_0} = 0$.

✓ $x$ is compatible with $(u, v)$ and $(u, v)$ is a feasible payoff vector.

## Definition

The feasible outcome $((u, v); x)$ is pairwise-stable if whenever $x_{ik} = 0$, $u_{im} + v_{lk} \geq \alpha_{ik}$ for all $i$'s partners $m$ and all $k$'s partners $l$. (or equivalently $u_i + v_k \geq \alpha_{ik}$, where

$u_i = \min\{u_{ik}\}$ for $k \in C(i, x)$ and $v_k = \min\{v_{ik}\}$ for $i \in C(k, x)$

# $M_1$: Example 3

$$\begin{array}{ccc|c|c|}
 & & & s_1 = 1 & s_2 = 2 \\
 & & & w_1 & w_2 \\
\cline{4-5}
r_1 = 2 & f_1 & & 3 & 2 \\
\cline{4-5}
r_2 = 2 & f_2 & & 3 & 3 \\
\cline{4-5}
\end{array}$$

- Let $x_{11} = x_{12} = x_{22} = 1$ and $x_{21} = 0$. ($f_2$ one unfilled position)
- Let $u_{11} = u_{12} = u_{22} = 1$, $u_{2w_0} = 0$, $v_{11} = 2$, $v_{12} = 1$, $v_{22} = 2$.
- $2 = u_{2w_0} + v_{11} < 3 \Rightarrow ((u, v); x)$ is not pairwise-stable:

$f_2$ offers $2 + \varepsilon > v_{11}$, with $0 < \varepsilon < 1$ to $w_1$ and gets $1 - \varepsilon$.

- There is another optimal matching:

$x' = \{(f_2, w_1), (f_2, w_2), (f_1, w_2), (f_1, w_0)\} \Rightarrow w_A(F \cup W) = 8$.

- The characteristic function is: $w_A(f_i) = w_A(w_k) = 0$,

$w_A(f_1, w_1) = 3, w_A(f_1, w_2) = 2,\ w_A(f_2, w_1) = 3,\ w_A(f_2, w_2) = 3$

$w_A(f_1, f_2, w_1) = 3$,

$w_A(f_1, f_2, w_2) = w_A(f_1, w_1, w_2) = 5, w_A(f_2, w_1, w_2) = 6$

Then $(U_1, U_2; V_1, V_2) = (2, 1; 2, 3)$ is in the core.

The set of pairwise-stable payoffs does not coincide with the core.

# $M_1$: Pairwise-stability

## Definition

The feasible matching $x$ is optimal if, for all feasible matching $x'$,

$$\sum_{(i,k)\in F\times W} \alpha_{ik} \cdot x_{ik} \geq \sum_{(i,k)\in F\times W} \alpha_{ik} \cdot x'_{ik}.$$

## Fact

*If $((u,v);x)$ is pairwise-stable, then $x$ is an optimal matching.*

## Theorem

*The set of pairwise-stable outcomes for $M_1(\alpha)$ is nonempty.*

# $M_1$: Example 3

| | | $s_1 = 1$ | $s_2 = 2$ |
|---|---|---|---|
| | | $w_1$ | $w_2$ |
| $r_1 = 2$ | $f_1$ | **3** | **2** |
| $r_2 = 2$ | $f_2$ | 3 | **3** |

Fix an optimal matching $(x_{11} = x_{12} = x_{22} = 1 = x_{20})$ and define the related one-to-one assignment market:

| | $w_1^1$ | $w_2^1$ | $w_2^2$ |
|---|---|---|---|
| $f_1^1$ | **3** | 0 | 0 |
| $f_1^2$ | 0 | **2** | 0 |
| $f_2^1$ | 3 | 0 | **3** |
| $f_2^2$ | 3 | 0 | 0 |

A core-element of the one-to one assignment game gives a pairwise-stable outcome of $M_1$, for instance:

$$(2,2,3,0;1,0,0) \quad \longrightarrow (u_{11}, u_{12}, u_{22}, u_{20}; v_{11}, v_{12}, v_{22}, v_{20}) = (2,2,3,0;1,0,0,0)$$

$$(0,0,0,0;3,2,3) \quad \longrightarrow (u_{11}, u_{12}, u_{22}, u_{20}; v_{11}, v_{12}, v_{22}, v_{20}) = (0,0,0,0;3,2,3,0)$$

# $M_1$: Optimal pairwise-stable outcomes

**Theorem**

*There exists at least one $F$-optimal pairwise-stable outcome and one $W$-optimal pairwise-stable outcome for $M_1(\alpha)$.*

Take $x$ an optimal matching, if $((\overline{u}', \underline{v}'); \tilde{x})$ is the $F$-optimal stable outcome of a related one-to-one assignment game, consider the related pairwise-stable outcome for $M_1(\alpha)$: $((\overline{u}, \underline{v}); x)$. This is $F$-optimal for $M_1(\alpha)$: for all pairwise-stable outcome $((u, v); x')$,

$$\sum_{k \in W} \overline{u}_{ik} x_{ik} \geq \sum_{k \in W} u_{ik} x'_{ik} \text{ for all } i \in F.$$

✓ Any algorithm to compute the optimal stable outcomes of a simple assignment game can be used to obtain the optimal stable outcomes of the multiple partners game.

# $M_1$: Competitive equilibria (Sotomayor, 2007)

Let us now think of buyers and sellers instead of firms and workers.

**Definition**

Given $(B, Q, A, r, s)$, the feasible outcome $((u, p); \mu)$ is a competitive equilibrium iff

1. For all $b \in B$, if $\mu(b) = S$, then $S \in D_b(p)$,
2. For all $q \in Q$ unsold, $p_q = 0$.

✓In a competitive equilibrium, every seller sells all his items at the same price. If a seller has two identical objects, $q$ and $q'$ and $p_q > p_{q'}$, then no buyer will demand a set of objects $S$ that contain object $q$ (since by replacing by $q'$ will obtain a more preferable set of objects). Then $q$ would remain unsold with a positive price, in contradiction with the definition of competitive price outcome.
✓ This is due to the assumption of the model under which no buyer is interested in acquiring more than one item of a given seller.

# $M_1$: Competitive equilibria

- Every competitive-equilibrium outcome is a pairwise-stable outcome.
- A pairwise equilibria outcome where the sold objects of a same seller have the same price is a competitive-equilibrium outcome.
- Given a pairwise stable outcome $((u, v), \mu)$, define $v'_{pq} = \min_{q \in \mu(p)} v_{pq}$ and $u'$ the corresponding payoff for the buyers. Then $((u, v), \mu)$ is a competitive-equilibrium payoff.

| | | $s_1 = 1$ | $s_2 = 2$ |
|---|---|---|---|
| | | $w_1$ | $w_2$ |
| $r_1 = 2$ | $f_1$ | **3** | **2** |
| $r_2 = 2$ | $f_2$ | 3 | **3** |

$$(u_{11}, u_{12}, u_{22}, u_{20}; v_{11}, v_{12}, v_{22}, v_{20}) = (2, 2, 3, 0; 1, 0, 0, 0)$$

$$(u_{11}, u_{12}, u_{22}, u_{20}; v_{11}, v_{12}, v_{22}, v_{20}) = (0, 0, 0, 0; 3, 2, 3, 0) \rightarrow (0, 0, 1, 0; 3, 2, 2, 0$$

# $M_1$: Competitive equilibria

- In Sotomayor (1999) it is proved the lattice structure of the set of pairwise-stable payoffs.
- By the above procedure, this structure is inherited by the set of competitive equilibria payoffs.
- Hence, there exists a buyers-optimal competitive equilibria payoff vector and a sellers-optimal competitive equilibria payoff vector.

# $M_1$: The core

An outcome specifies for each agent a set of payments made by the group of agents matched to him. Thus an agent's payoff is the sum of these payments. We now look directly at the total payoff of each agent (there is a loss of information).

**Definition**

A feasible payoff is $((U, V); x)$, where $x$ is a feasible matching, $U \in \mathbb{R}_+^F$, $V \in \mathbb{R}_+^W$ and
i) $U_i = 0$ if $i$ unmatched; $V_k = 0$ if $k$ unmatched,
ii) $\sum_{i\in F} U_i + \sum_{k\in W} V_k \leq \sum_{(i,k)\in F\times W} \alpha_{ik} x_{ik}$.

**Definition**

The feasible payoff $((U, V); x)$ is in the core if there are no subsets $R \subseteq F$, $S \subseteq W$ and a feasible matching $x'$ such that

$$\sum_{i\in R} U_i + \sum_{k\in S} V_k < \sum_{(i,k)\in R\times S} \alpha_{ik} x'_{ik}.$$

# $M_1$: The core

✓ Coalitional rationality for buyer-seller pairs does not suffice to describe the core.

✓ A market with one firm and three workers.

| | | $s_1 = 1$ | $s_2 = 1$ | $s_3 = 1$ |
|---|---|---|---|---|
| | | $w_1$ | $w_2$ | $w_3$ |
| $r_1 = 2$ | $f_1$ | 1 | 2 | 3 |

- The feasible outcome $((U, V); x)$ where $U = 1$, $V = (0, 1, 3)$ and $x = (0, 1, 1)$ is blocked by $R = \{f_1\}$, $S = \{w_1, w_2\}$ and the matching $x' = (1, 1, 0)$.
- But there are no blocking pairs since $U_1 + V_k \geq \alpha_{1k}$ for all $k$.

**Theorem**

*Every pairwise-stable outcome $((u, v); x)$ for $M_1(\alpha)$ gives a payoff vector $((U, V); x)$ in the core of the game generated by this market: $\sum_{f \in S} U_f + \sum_{w \in R} V_w \geq w_A(S \cup R)$. Hence, the core is nonempty.*

# References

- Sotomayor, M. *The multiple partners game.* In: Majumdar, M. (ed.) Equilibrium and Dynamics: Essays in honor to David Gale, 1992.
- Sotomayor, M. *The lattice structure of the set of stable outcomes of the multiple partners assignment game.* IJGT, 1999.
- Sotomayor, M. *Connecting the cooperative and competitive structures of the multiple partners assignment game.* JET, 2006.
- Sotomayor, M. *A note on the multiple partners assignment game.* JME, 2009.

# Multiple-partners assignment market: Model 2 (Thompson, 1981; Crawford and Knoer, 1981; Sotomayor, 2002)

- Let $F$ be a finite set of firms, $W$ a finite set of workers and for each $(f, w) \in F \times W$, $a_{fw}$ represents the amount of income the pair can generate.
- The capacity of each agent is not the number of different partnerships he can establish but the number of units of work he supplies or demands. Let $p_i$ be the capacity of firm $i \in F$ and $q_j$ the capacity of worker $j \in W$.
- In Operations Research, finding and optimal assignment to this situation is known as the transportation problem.

$$\begin{array}{ll} \max & \sum_{F\times W} x_{ij} a_{ij} \\ \text{where} & \sum_{j\in W} x_{ij} \leq p_i, \text{ for all } i \in F, \\ & \sum_{i\in F} x_{ij} \leq q_j, \text{ for all } j \in W. \\ & x_{ij} \geq 0, \text{ for all } (i,j) \in F \times W. \end{array}$$

If $p_i, q_j \in \mathbb{Z}$, there exists integer solution $x = (x_{ij})$ (Dantzing,1963).

# $M_2$: Solutions to the dual linear problem

- The dual linear problem is:

$$\begin{array}{ll} \min & \sum_{i\in F} p_i y_i + \sum_{j\in W} q_j z_j \\ \text{where} & y_i + z_j \geq a_{ij}, \text{ for all } (i,j) \in F \times W, \\ & y_i \geq 0, z_j \geq 0, \text{ for all } (i,j) \in F \times W. \end{array}$$

- Given a solution $(y, z)$ to the dual problem, the payoff vector $(u, v)$ where $u_i = p_i y_i$ for all $i \in F$ and $v_j = q_j z_j$ for all $j \in W$, belongs to the core of the related assignment game.
- In this vector, each firm pays equally each unit of labour (even though they correspond to different workers) and each worker receives the same payment for each unit of labour (even though they correspond to different firms).

**Theorem**

*The core of the multiple-partner assignment game $M_2$ is non-empty*

# $M_2$: Differences with the assignment game

- The core strictly contains the set of solutions of the dual problem.
  For instance, in a market with one firm $f_1$ with capacity $r_1 = 2$, one worker $w_1$ with capacity $s_1 = 1$ and $a_{11} = 4$.
  The characteristic function is $w_A(f_1) = w_A(w_1) = 0$, $w_A(f_1, w_1) = 4$.
  The core is $\{(u, 4-u) \mid 0 \leq u \leq 4\}$ but the only solution to the dual problem is $(0, 4)$.
- Inside the core there is no oposition of interest between the two sides of the market and the core is not a lattice.

|  |  | $s_1 = 1$ | $s_2 = 1$ |
|---|---|---|---|
|  |  | $w_1$ | $w_2$ |
| $r_1 = 2$ | $f_1$ | 3 | 3 |

$w_A(f, w_1) = w_A(f, w_2) = 3, w_A(f, w_1, w_2) = 6$

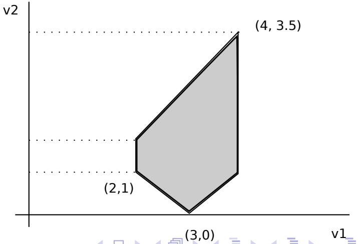
$(u; v) = (5; 1, 0), (u'; v') = (4; 0, 2) \in C(w_A)$ but

$(u \vee u', v \wedge v') = (5; 0, 0) \notin C(w_A)$

# $M_2$: Existence of optimal core elements for each sector

- It is an open problem the existence of a core element that is optimal for each side of the market.
- There may not be a worst core element for one side of the market.

|  |  | $s_1 = 1$ | $s_2 = 3$ |
|---|---|---|---|
|  |  | $w_1$ | $w_2$ |
| $r_1 = 2$ | $f_1$ | 4 | 1 |
| $r_2 = 2$ | $f_2$ | 4.5 | 1.5 |



# $M_2$: The many-to-one case

- All agents on one side (let us say the workers) have capacity 1.
- Then, there exists an optimal core allocation for each side of the market (which is the worst one for the opposite side).
- But the core does not have a lattice structure

# References

- Kaneko, M. *On the core and competitive equilibria of a market with indivisible goods*, Naval Research Logistics Quarterly, 1976.
- Thompson, G.L. *Computing the core of a market game*, 1980.
- Crawford V. and Knoer E.M. *Job matching with heterogeneous firms and workers.* Econometrica, 1981.
- Sánchez-Soriano, J. et al. *On the core of transportation games.* MSS, 2001.
- Sotomayor, M. *A labor market with heterogeneous firms and workers.* IJGT, 2002.
- Camiña, E. *A generalized assignment game.* MSS, 2006.
- Jaume, D.,Masso, J and Neme, A. *The multiple-partners assignment game with heterogeneous sells and multi-unit demands: competitive equilibria.* MMOR, 2012.

Thank you!