Parameterized complexity of some stable matching problems

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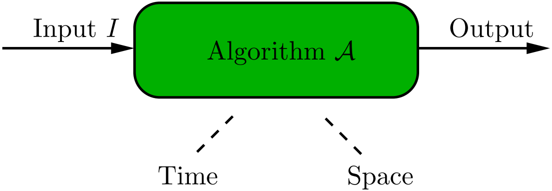
Outline

- Algorithms and complexity: the classical view
- Parameterized complexity:
  - Fixed-Parameter Tractability (FPT)
  - W[1]-hardness
- Parameterizing hard variants of Stable Matching:
  - Stable Matching with Ties and Incomplete Lists
  - Egalitarian, Minimum Regret, and Sex-Equal Stable Matching
  - Hospitals/Residents with Couples: different variants
- Future directions?

Algorithms and complexity



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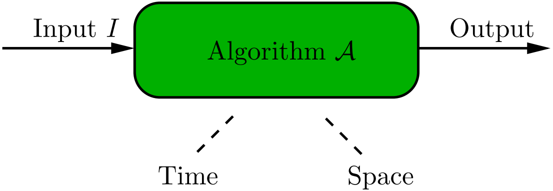

Running time of $\mathcal{A}$:

- Depends on the input I.
- Measuring the complexity of I: size of I.
- $T(n) =$ maximum number of steps on any input of size $\leq n$.
- Efficient algorithms: $T(n) = n$, $T(n) = n^2$, $T(n) = n^3$, ...
  $\longrightarrow$ $T(n)$ should be a polynomial (having fixed degree).
  $\Longrightarrow$ **P**: Polynomial-time solvable problems.

Hard problems

NP-hard problems:

- No polynomial-time algorithm is known for them.
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What to do with hard problems in practice?

- “Best effort”: exponential-time algorithms.
- Approximation: finding a sub-optimal solution fast, with a quality guarantee.
  $\longrightarrow$ we want to maximize the value of an objective function f
  $\longrightarrow$ α-approximation: $f(\text{Output}) \geq \dfrac{\text{OPT}}{\alpha}$
- Heuristics: finding a sub-optimal solution fast, no guarantee.
- Parameterized complexity!

Parameterized complexity

Framework for dealing with hard problems [Downey & Fellows, 1999]

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- Motivation: if the parameter k is small in practice, an FPT algorithm can be efficient.

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Vertex Cover

Input: an undirected graph G and an integer k.
Task: find a vertex cover of size at most k.

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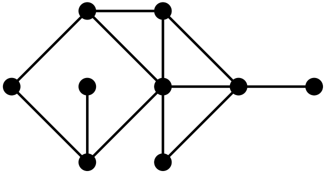

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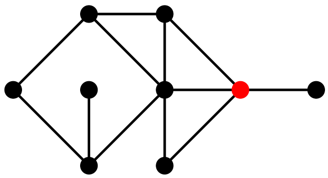

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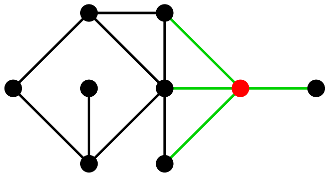

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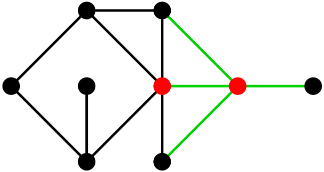

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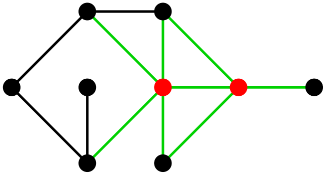

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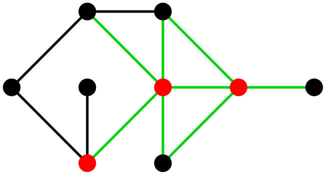

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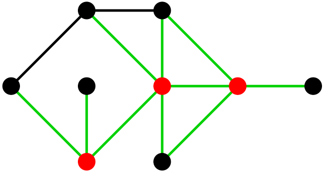

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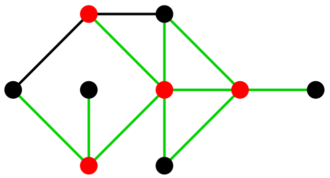

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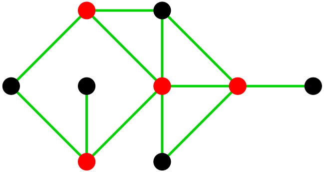

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- Can we do better?
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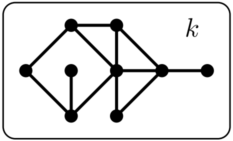
$\Longrightarrow$ algorithm:

1. Let $S = \emptyset$.
2. While there is an uncovered edge xy:
   if $k = 0$, then reject;
   if $k > 0$, then branch into two directions:
   $\rightarrow$ branch 1: $S := S \cup \{x\}$, $k = k - 1$.
   $\rightarrow$ branch 2: $S := S \cup \{y\}$, $k = k - 1$.
3. Output S.

k-Vertex Cover – an $O(2^k kn)$ algorithm

Bounded search tree algorithm:



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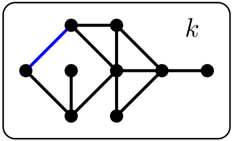

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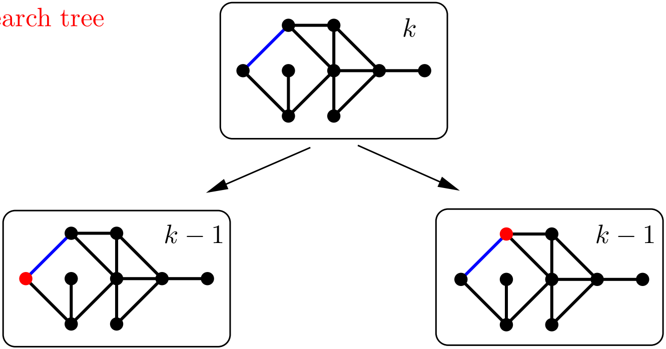

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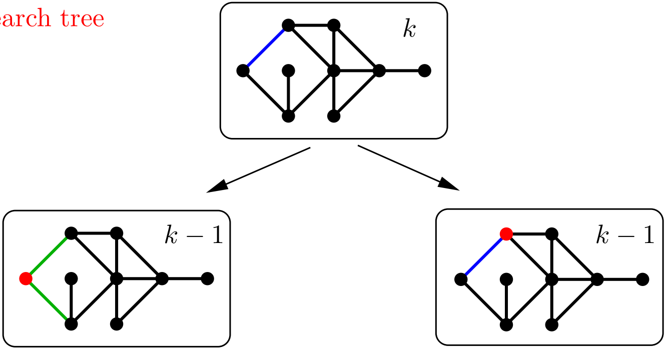

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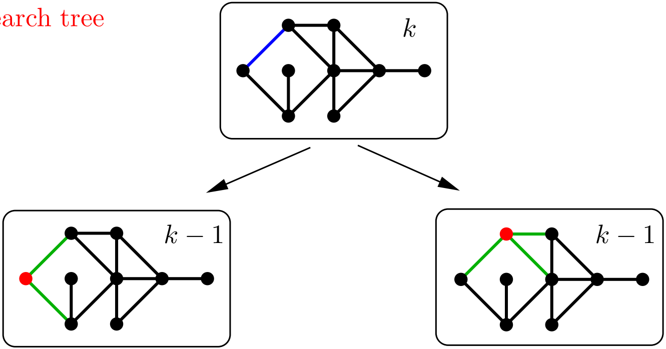

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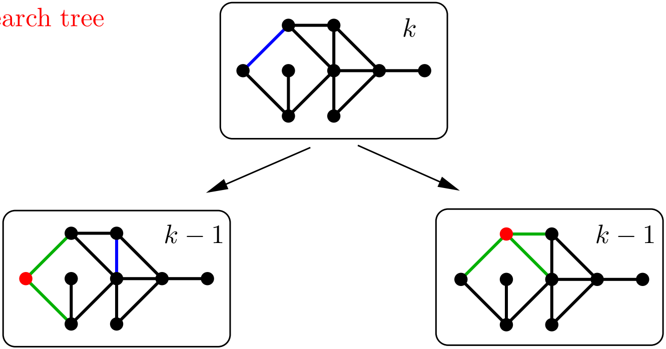

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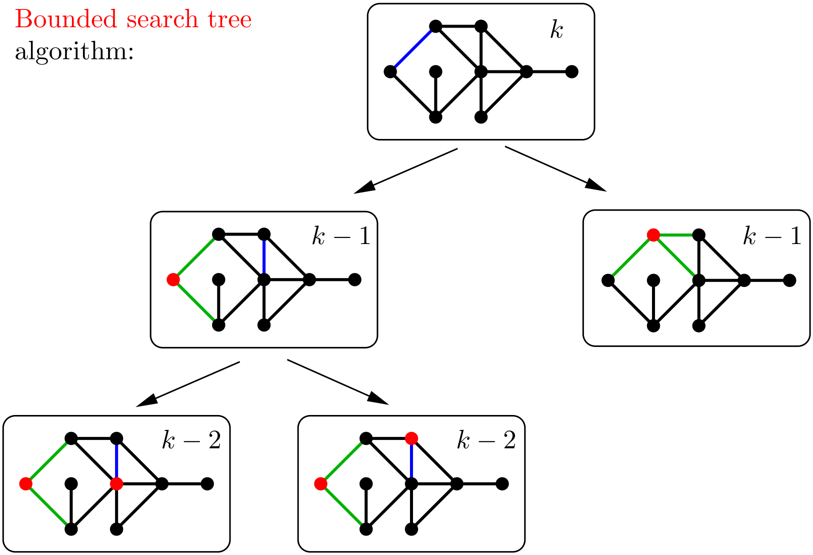

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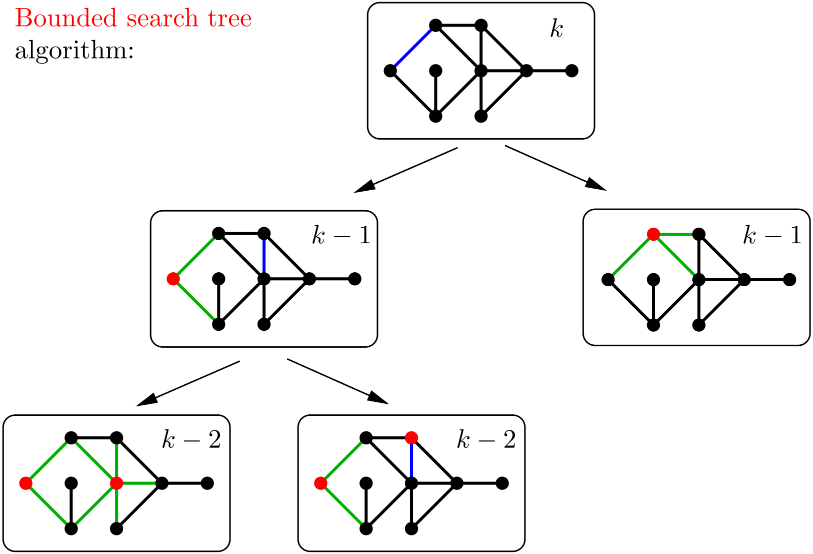

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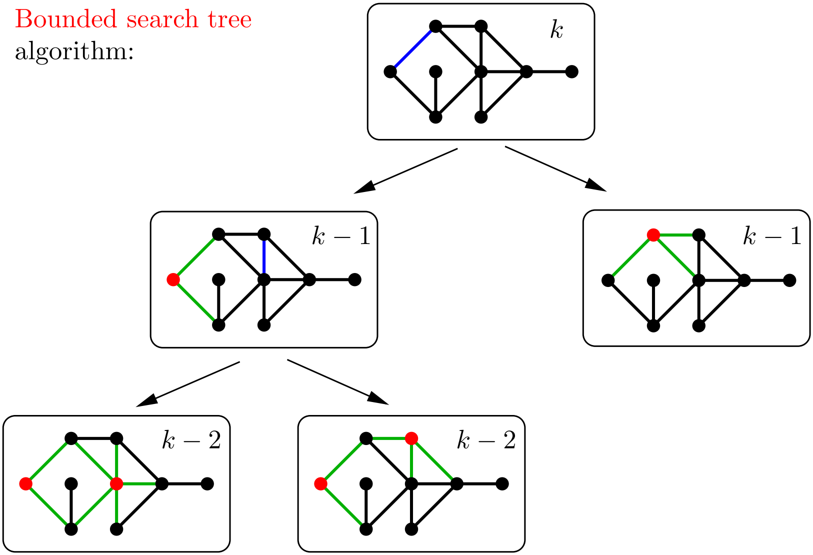

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- Each branching decreases the parameter.
  $\Longrightarrow$ Search tree has depth at most k
  $\Longrightarrow$ at most 2^k leaves.
- Computations at one node: $O(|E(G)|) = O(kn)$ time.
- Overall running time: $O(2^k kn)$
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Currently the fastest algorithm: $O(1.2738^k + kn)$ [Chen et al.]

$\Longrightarrow$ Vertex Cover is solvable for $n = 10^6$ and $k = 40$.

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- Can we get k out of the exponent?
  Is Clique FPT if the parameter is k?
  $\Rightarrow$ no FPT-algorithm is known... But can we prove it?

Parameterized hardness

W[1]-hardness:

- Analogous to NP-hardness.
- Hardness hierarchy: $\mathrm{FPT} \subseteq \underbrace{W[1] \subseteq W[2] \subseteq \cdots \subseteq WP}_{\text{intractable classes}}$
- W[1]-hard problems are unlikely to admit an FPT-algorithm.
  An FPT-algorithm for a W[1]-hard problem would yield an FPT-algorithm for *all* problems in W[1].
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$\Longrightarrow$ $O(n^k)$ seems optimal.

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FPT is closed under parameterized reductions:
If Q can be reduced to Q', and Q' is FPT, then Q is FPT as well.

Parameterized reductions: example I.

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- G' is the complement of G: $\forall e : e \in E(G') \iff e \notin E(G)$;
- $k' = k$.

Is this an FPT-reduction?

- G has a clique of size $k \iff G'$ has k independent vertices. $\checkmark$
- The reduction runs in FPT time. $\checkmark$
- k' is a function of k only.

Parameterized reductions: example I.

k-INDEPENDENT SET

Input: an undirected graph G.
Parameter: an integer k.
Question: is there an independent set of size k in G?

An *independent set* is a set of pairwise non-adjacent vertices.

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Parameterized reductions: example II.

A reduction from k-Independent Set to k-Vertex Cover:

$(G, k) \longrightarrow (G', k')$ where

- $G' = G$ and
- $k' = n - k$ $(n = |V(G)|)$.

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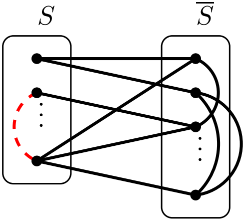
  S is an independent set in $G \iff \bar{S} = V(G) \setminus S$ is a vertex cover.



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Proving W[1]-hardness

How to prove W[1]-hardness?

Theorem.

If Q is W[1]-hard, and Q can be FPT-reduced to some problem Q', then Q' is W[1]-hard as well.

$\Longrightarrow$ We can prove W[1]-hardness of Q' by giving an FPT-reduction from any known W[1]-hard problem.

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Example:

- We know that k-Clique is W[1]-hard.
- We just gave an FPT-reduction from k-Clique to k-Independent Set.

$\Longrightarrow$ We proved that k-Independent Set is W[1]-hard.

Many parameters

Extension of the model: multiple parameters.

- Each input I has multiple parameters $k_1, k_2, \ldots, k_d \in \mathbb{N}$.
- Easy to extend the notation.
  An algorithm is FPT with combined parameters $(k_1, \ldots, k_d)$, if it runs in time

$$f(k_1, \ldots, k_d)|I|^{O(1)} \text{ for some function } f.$$

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- Very useful in practice!
  There can be many important parameters in a problem.
- Multi-dimensional view on the complexity of the problem
  $\Longrightarrow$ yields a more detailed insight.

How to choose the parameter?

A good parameter ...

- has small value in practice,
- makes the problem FPT.

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Typical parameters:

- size of the solution
- some natural, simple, problem-specific property of the input
- distance from triviality
  $\longrightarrow$ Which special cases are easy?
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Multivariate analysis: the more parameters we examine, the more knowledge we obtain about the problem.

Hard variants of some stable matching problems

NP-hard problems for which the parameterized complexity has been studied:

- Maximum Stable Matching with Ties and Incomplete Lists
- Minimum Regret Stable Matching
  Egalitarian Stable Matching
  Sex-Equal Stable Matching
- Hospitals/Residents with Couples (HRC)
  Matching with Couples
  Special HRC with master list
- Socially stable matchings for Hospitals/Residents
- Housing Markets with Duplicate Houses

SMTI

Stable Matching with Ties and Incomplete Lists:

- Input: A set W of women and a set U of men, and a set $\mathcal{L} = \{L_p \mid p \in W \cup U\}$ of preference lists.
- L_p may be incomplete: contains only acceptable partners.
  L_p may contain ties: possible partners equally good for p.

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- L_p may be incomplete: contains only acceptable partners. L_p may contain ties: possible partners equally good for p.
- A matching M is a set of acceptable man-women pairs, containing each person at most once.
  $\longrightarrow$ It is a matching in the underlying bipartite graph.
  $\longrightarrow$ $M(p)$ is the person matched to p by M.
- $(w, m) \in W \times U$ is a blocking pair w.r.t. a matching M, if
  - w is unmatched, or strictly prefers m to $M(w)$; and
  - m is unmatched, or strictly prefers w to $M(m)$.
- M is stable $\iff$ there is no blocking pair for M.

MaxSMTI

The Gale-Shapley algorithm can find *some* stable matching in linear time. But what about its size?

MaxSMTI

Input: An SMTI instance $I = (W, U, \mathcal{L})$ as described above.
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Easy special cases: no ties OR preference lists are complete:
$\Longrightarrow$ all stable matchings have the same size,
$\Longrightarrow$ the Gale-Shapley algorithm is optimal.

Parameterized complexity of MaxSMTI

Possible parameters:

- the maximum length of ties;
  Recall: MaxSMTI is NP-hard, even if each tie has length 2!
  $\Longrightarrow$ not a good parameter.
- the number of ties;
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MaxSMTI is W[1]-hard if the parameter is the number of ties, even if ties are only on the women's side.

FPT algorithm for MAXSMTI (par.: $\sum$ length of ties)

FPT algorithm for MAXSMTI:

1. Break ties in all possible ways for the given SMTI instance I. Let $I_1, I_2, \ldots, I_t$ be the obtained instances (without ties).
2. For each I_j, $j = 1, \ldots, t$, compute a stable matching M_i.
3. Output the largest among $M_1, M_2, \ldots, M_t$.

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Any stable matching M for I is also stable for *some* instance obtained by breaking ties.

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- Suppose T is a tie in w's preference list.
- If w is matched to some $m \in T$, then we break T such that m becomes the most preferred men in T.
  Otherwise, we can break T arbitrarily.
  $\Longrightarrow$ No blocking pair can appear, M remains stable.

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We can break ties in at most $L!$ possible ways. (L: $\sum$ length of ties)
$\Longrightarrow$ running time: $O(L!|I|)$
$\Longrightarrow$ FPT with parameter L.

Fair stable matchings

Maximality vs. fairness?

- We want a stable matching that is *fair* (not necessarily maximal).

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- Many different notions are in use.
- The *cost* of a person p in a matching M:

$$c_M(p) = \begin{cases} \text{the rank of } M(p) \text{ in } L_p, & \text{if } p \text{ is matched in } M; \\ |L_p| + 1, & \text{if } p \text{ is unmatched in } M. \end{cases}$$

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- M is egalitarian, if it minimizes $\sum c_M(p)$.
  (≈ optimizing total happiness)
- M is minimum regret, if it minimizes $\max c_M(p)$.
  (≈ optimizing on the least satisfied person)
- M is sex-equal, if it minimizes

$$\delta(M) = \left| \sum_{m \in U} c_M(m) - \sum_{w \in W} c_M(w) \right|.$$

$\longrightarrow \delta(M) \approx$ difference between men's and women's happiness.

Egalitarian and minimum regret stable matchings

Egal SMTI

Input: an SMTI instance I.
Task: find an egalitarian stable matching for I.

MinReg SMTI

Input: an SMTI instance I.
Task: find a minimum regret stable matching for I.

Complexity:

- If no ties are involved $\Longrightarrow$ both can be solved in polynomial time. [Irving et al.], [Gusfield]
- If ties can occur, then both problems become NP-hard, and even hard to approximate. [Halldórsson et al.]

Parameterizing Egal SMTI and MinReg STMI

Possible parameters:

- the maximum length of ties;
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Theorem [Manlove et al.]

If P $\neq$ NP and $\varepsilon > 0$, then no polynomial time algorithm can approximate the Egal SMTI or the MinReg SMTI problem within a factor of $N^{1-\varepsilon}$, where N is the number of men, even if ties are only present on women's side, and each tie has length 2.

$\Longrightarrow$ maximum length of ties: not a good choice.

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Simple FPT-algorithm:

1. Break ties in all possible ways.
2. For each obtained instance, apply the standard poly-time algorithm for finding an egalitarian or a minimum regret matching.

Important: break ties in a cost-preserving way!
$\longrightarrow$ use explicit *ranking functions* (instead of precedence lists).

FPT-inapproximability

FPT-approximation algorithm:

- Approximates a given optimization problem within some guaranteed factor, and
- runs in FPT time (instead of polynomial time).

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Theorem [D. Marx and I. S.]

If $\varepsilon > 0$ and W[1] $\neq$ FPT, then there is no FPT algorithm with the parameter being the number of ties, that can approximate MinReg SMTI or Egal SMTI within a factor of $N^{1-\varepsilon}$, even if ties are only present on women's side.

Sex-equal stable matchings

Sex-equal SMI

Input: an SMI instance I and an integer δ.
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- We assume that preference lists are strictly ordered: an SMI instance is an SMTI instance without ties.
- Recall: the sex-equality measure of a matching M is

$$\delta(M) = \left| \sum_{m \in U} c_M(m) - \sum_{w \in W} c_M(w) \right| .$$

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Complexity:

- NP-hard, even if the preference lists are complete. [Kato]
- If ties can occur, then NP-hard to approximate within a factor of εN for some $\varepsilon > 0$. [Halldórsson et al.]

Parameterizing Sex-equal stable matchings

Parameters examined by McDermid and Irving:

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Sex-equal SMI is NP-hard, even if $\delta = 0$ and all preference lists are of length at most 3.

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SEX-EQUAL SMI is polynomial-time solvable if the preference lists of women (or men) are of length at most 2.

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$\longrightarrow$ Not parameterized complexity in the strict sense.
$\longrightarrow$ Seek for parameters which, when small, make the problem easy.

The Hospitals/Residents problem

Hospitals/Residents: many-to-one version of Stable Matching.

Problem instance for Hospitals/Residents.

- agents: a set R of residents and a set H of hospitals
- a capacity $f(h)$ for each $h \in H$, giving the number of open jobs
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Task: find a stable assignment $M : R \to H$, respecting capacities.

- M is stable $\Leftrightarrow$ no blocking pair exists for M
- $(r, h) \in R \times H$ is a blocking pair for M, if they are both beneficial for each other w.r.t. M, meaning that
  - r is unemployed or prefers h to $M(r)$, and
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The HOSPITALS/RESIDENTS problem

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Complexity: solvable by an extension of the Gale-Shapley algorithm.

Hospitals/Residents with Couples (HRC)

Problem instance for HRC

- a set H of hospitals with a capacity function f
- a set C of couples, each $c \in C$ is a pair (c_1, c_2) of residents
- a set S of single residents
- strict preference lists (denoted by L)
  - hospitals rank acceptable residents
  - singles rank acceptable hospitals
  - couples rank acceptable pairs of hospitals
    example: $L(c) : (h_1, h_1), (h_2, h_3)$

Motivation:

- NRMP program in the US: assigning residents to hospitals
- US Navy detailing process

Hospitals/Residents with Couples (HRC)

Stability under HRC:

- assignment M is stable $\Leftrightarrow$ no blocking pair exists for M
- blocking "pair" for M:
  - **(a)** (s, h) where s is a single and h a hospital that are beneficial for each other w.r.t. M, or
  - **(b)** a couple $c = (c_1, c_2)$ and a pair of hospitals (h_1, h_2) such that
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Complexity of HRC:

- It is NP-hard to decide whether a stable assignment exists. [Ronn]
- Stable assignments of various sizes may exist.
  The size of an assignment: the number of residents having a job.

Parameterized complexity of HRC

Parameter: the number $|C|$ of couples

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- HRC is solvable in $|I|^{O(|C|)}$ time:
  1. We try each possible way to fix the assignment on each couple: $\approx |H|^{2|C|}$ possibilities.
  2. Assign as many singles as possible to the remaining jobs: easy!

$\Longrightarrow$ Polynomial-time solvable for each fixed $|C|$.
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Theorem

The existence version of Hospitals/Residents with Couples problem is W[1]-hard with parameter $|C|$.

Local Improvement for HRC

Suppose we already have a stable assignment, possibly not maximal.
Question: can we improve it efficiently?

- Such an algorithm would be extremely useful in practice!
- General form: as hard as the original problem.
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Local Improvement for HRC

Input: an instance I of HRC, a stable assignment M for I, and $\Delta \in \mathbb{N}$.
Task: find a stable assignment M' for I such that

1. M' is larger than M, and
2. M' differs from M only for at most Δ residents.

Local Improvement for HRC

Suppose we already have a stable assignment, possibly not maximal.
Question: can we improve it efficiently?

- Such an algorithm would be extremely useful in practice!
- General form: as hard as the original problem.
- What if we only look for small modifications? → local search

Local Improvement for HRC

Input: an instance I of HRC, a stable assignment M for I, and $\Delta \in \mathbb{N}$.
Task: find a stable assignment M' for I such that

1. M' is larger than M, and
2. M' differs from M only for at most Δ residents.

Theorem [D. Marx and I. S.]

Local Improvement for HRC is FPT with parameters $(|C|, \Delta)$.

Matching with Couples

Simplification of HRC:

- We forget about the preferences.
- We aim for an acceptable assignment:
  each agent (single, couple, or hospital) must be assigned to an acceptable partner.
- Application in scheduling.

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Maximum Matching with Couples

Input: An instance $I = (H, S, C, A, f)$ of MMC.

- H, S, C, f: as before (hospitals, singles, couples, capacities)
- A: list of acceptable partners for each agent.

Task: Find an assignment for I having maximum size.

Complexity of Maximum Matching with Couples

Classical complexity:

Theorem [Glass, Kellerer],[Biró,McDermid]

Maximum Matching with Couples is NP-hard, even if each capacity is 2.

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Parameterized complexity with parameter $|C|$:

Theorem [D. Marx and I. S.]

Maximum Matching with Couples can be solved in randomized FPT time, if the parameter is the number $|C|$ of couples.

Special HRC with master list

Special HRC: simplifications based on real-world applications.

- Each resident has a score, yielding a master list for them $\Longrightarrow$ hospitals rank the residents according to their scores.
- Hospital pairs can be compatible or not.
- Preference list of a couple $c = (a, b)$:
  - a and b have individual preference lists.
  - $(h_1, h_2) \in L(c) \iff$ (i) $h_1 \in L(a)$ and $h_2 \in L(b)$, and (ii) h_1 and h_2 are compatible.
  - Responsive preferences: if $h_1 \succ_a h_3$ and $h_2 \succ_b h_4$, then $(h_1, h_2) \succ_c (h_3, h_4)$.
- Assumption: strict preferences.

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Special HRC

Input: a Special HRC instance I as described above.
Task: find a stable assignment for I.

Complexity of Special HRC

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Special HRC can be solved in FPT time, if the parameter is the number $|C|$ of couples.

FPT-algorithm for SPECIAL HRC

1. Preprocessing phase: finds an initial assignment M_0.
   - Order the single residents decreasingly by their score, and assign each one to its most preferred hospital still available.
   - For each resident r, delete all hospitals h from $L(r)$ for which M_0 assigns $c(h)$ applicants better than r.
     $\longrightarrow$ if r is a member of a couple c, update the list $L(c)$ as well.

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2. While there is an unassigned couple:
   - (a, b): the couple where a is the best member of any couple.
   - Prune $L(a, b)$ to contain at most $2|C|$ entries.
     Crucial step: must be done safely!
     $\Longrightarrow$ We only remove irrelevant entries.
   - Try each possibility for assigning (a, b), and update the capacities.

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Running time: $(2|C|)^{|C|} \cdot |I|^{O(1)}$ $\Longrightarrow$ FPT with parameter $|C|$. ✓

Conclusion

Take home message.

Parameterized complexity is a powerful and rich framework to deal with computationally hard problems.

Further research:

- Plenty of work to do!
  Parameterized results related to stable matchings: <10 papers.
  Parameterized results in computational social choice: much more.
- We need more FPT results
  $\Longrightarrow$ find good parameters and tractable models!
- Multiple parameters $\Longrightarrow$ more detailed insight.
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Thank you!