Axiomatic Foundations of Voting Theory (part I)

William S. Zwicker
Mathematics Department, Union College

Computational Social Choice Summer School
San Sebastian, Spain
18-22 July 2016
COST IC1205
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1) Intro: Three voting rules

• Election with 3 candidates a, b, c for mayor of a town
• 303 voters
• Each voter casts a ballot
1) Intro: Three voting rules

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*complete information about an election*
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\[
P_1
\begin{array}{ccc}
102 & 101 & 100 \\
 a & b & c \\
b & c & b \\
c & a & a \\
\end{array}
\]

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\(P_1\) is a **voting situation**, not a profile (**incomplete info**)
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- A **plurality ballot** specifies a single most preferred candidate

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- Plurality voting rule: winner = candidate with most votes

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\(P_1\) is a **voting situation**, not a profile (**incomplete** info)

- Plurality winner for \(P_1\) is a
- No **majority** winner exists
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\(P_1\) is a voting situation, not a profile (incomplete info)
- Plurality winner for \(P_1\) is \(a\)
- No majority winner exists
- For 3 or more candidates, plurality \(\neq\) majority
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\(P_1\) is a **voting situation**, not a profile *(incomplete info)*

- Plurality winner for \(P_1\) is *a*
- Say we used a different voting rule – one using info in full ranking . . . ?
- **Who should win?**
1) Intro: Three voting rules

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- Using most rules, $b$ wins
1) Intro: Three voting rules

- Plurality rule is common in RW
- Elect US senator from NY State
- 3-way 1980 vote

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- New York sits on Left (US terms)

\[ P_1 \]

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- Outcome was very parallel to $P_1$
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\[ P_1 \]

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\begin{array}{ccc}
102 & 101 & 100 \\
\text{a} & \text{b} & \text{c} \\
\text{b} & \text{c} & \text{b} \\
\text{c} & \text{a} & \text{a} \\
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- Outcome was very parallel to \( P_1 \)
- H and J split L vote; D’Amato won
- Such examples are major reason for opposition to plurality rule . . .
- . . . and interest in voting theory

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- Using most rules, b wins
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• We agreed . . . on almost nothing . . . *
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- Vote on Voting Rules
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- . . . *but note score for plurality is 0
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• Laslier’s article: “And the loser is ... Plurality Voting”
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• Laslier’s article: “And the loser is ... Plurality Voting”

• Which voting rule won?
• 2010 Meeting, Ch. du Baffy, Normandy

• We agreed . . . on almost nothing . . . *

• Vote on Voting Rules

• . . . *but note score for plurality is 0

• Laslier’s article: “And the loser is ... Plurality Voting”

• Which voting rule won?

• What question should you be asking me . . . ?
1) Intro: Three voting rules

\[ P_2 \]

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1) Intro: Three voting rules

- In profile $P_2$
  - 202 voters rank $a$ over $b$

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1) Intro: Three voting rules

- In profile $P_2$
  - 202 voters rank a over b
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\[
\begin{array}{cccc}
102 & 101 & 100 & 1 \\
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1) Intro: Three voting rules

- In profile $P_2$
  - 202 voters rank a over b
  - 102 rank b over a
  - $\text{Net}_{P_2}(a > b) = 202 - 102 = 100$
1) Intro: Three voting rules

- In profile $P_2$
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  - 102 rank b over a
  - $Net_{P_2}(a > b) = 202 - 102 = 100$

- We get a **weighted tournament** induced by the profile $P_2$

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• **Tournament:**
  - A graph in which the vertices are the candidates
  - For each two vertices, *either* $a \rightarrow b$ *or* $a \leftarrow b$ is an edge
1) Intro: Three voting rules

- In profile $P_2$
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**Tournament:**
- A graph in which the vertices are the candidates
- For each two vertices, *either* $a \rightarrow b$ *or* $a \leftarrow b$ is an edge

**Edge weights:**
- Assign $\text{Net}_{P_2}(a > b)$ to $a \rightarrow b$
1) Intro: Three voting rules

Pairwise Majority Preference
• $x >^\mu y$ means (strictly) more voters rank $x$ over $y$ than $y$ over $x$

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• $x >^\mu y$ means (strictly) more voters rank $x$ over $y$ than $y$ over $x$
• Equivalently, $\text{Net}_p(x>y) > 0$

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Pairwise Majority Preference

• $x \succ^\mu y$ means (strictly) more voters rank $x$ over $y$ than $y$ over $x$
• Equivalently, $\text{Net}_p(x>y) > 0$
• For $P_2$, we have $a \succ^\mu b \succ^\mu c \succ^\mu a$
  ➢ *majority cycle/Condorcet cycle*
  ➢ $\succ^\mu$ is *intransitive*
1) Intro: Three voting rules

Pairwise Majority Preference

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- For \( P_2 \), we have \( a >^\mu b >^\mu c >^\mu a \)
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- 3 BIG results of voting theory
  - majority cycles
  - Arrow’s impossibility Thm
  - Gibbard-Satterthwaite Thm

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**3 BIG results of voting theory**
  - *majority cycles*
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### Copeland Voting Rule
- Symmetric Copeland Score
  \[ \text{Cop}(x) = |\{y | x >^\mu y\}| - |\{y | y >^\mu x\}| \]
1) Intro: Three voting rules

**Pairwise Majority Preference**
- $x \succ y$ means (strictly) more voters rank $x$ over $y$ than $y$ over $x$
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**Copeland Voting Rule**
- Symmetric Copeland Score
  
  $\text{Cop}(x) = |\{y | x \succ y\}| - |\{y | y \succ x\}|$
- For $P_2$, $\text{Cop}(a) = 1 - 1 = 0$
1) Intro: Three voting rules

Pairwise Majority Preference
• $x >_\mu y$ means (strictly) more voters rank $x$ over $y$ than $y$ over $x$
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Copeland Voting Rule
• Symmetric Copeland Score
  $Cop(x) = |\{y | x >_\mu y\}| - |\{y | y >_\mu x\}|$
• For $P_2$, $Cop(a) = 1 - 1 = 0$
  $0 = Cop(b) = Cop(c)$  Tie!
1) Intro: Three voting rules

**Pairwise Majority Preference**
- $x \succ^\mu y$ means (strictly) more voters rank $x$ over $y$ than $y$ over $x$
- Equivalently, $\text{Net}_p(x>y) > 0$
- For $P_2$, we have $a \succ^\mu b \succ^\mu c \succ^\mu a$
  - *majority cycle/Condorcet cycle*
  - $\succ^\mu$ is *intransitive*
- 3 BIG results of voting theory
  - *majority cycles*
  - *Arrow’s impossibility Thm*
  - *Gibbard-Satterthwaite Thm*

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![Diagram](attachment:image.png)
1) Intro: Three voting rules

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![Diagram](image)

Cop’d ignores size of margins in pairwise wins/losses . . . *very indecisive*
1) Intro: Three voting rules

Pairwise Majority Preference
• x >\(\mu\) y means (strictly) more voters rank x over y than y over x
• Equivalently, Net\(_p\)(x>y) > 0
• For P\(_2\), we have \(a >\mu b >\mu c >\mu a\)
  ➢ *majority cycle/Condorcet cycle*
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Copeland Voting Rule
• *Symmetric* Copeland Score
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  \text{Cop}(x) = \left| \{y | x >\mu y\} \right| - \left| \{y | y >\mu x\} \right|
  \]
• For P\(_2\), Cop(a) = 1 – 1 = 0
  0 = Cop(b) = Cop(c) *Tie!*

\[
\begin{array}{c}
\text{b} \\
\text{102}
\end{array}
\begin{array}{c}
\text{a} \\
\text{100}
\end{array}
\begin{array}{c}
\text{c} \\
\text{100}
\end{array}
\]

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Copeland Voting Rule
• Symmetric Copeland Score
  $\text{Cop}(x) = |\{y | x >^\mu y\}| - |\{y | y >^\mu x\}|$

Exercise 1: Do other versions of Copeland score yield the same rule?
1) Intro: Three voting rules

**Borda Voting Rule**
- Symmetric Borda Score
  \[ \beta(x) = \sum_y \text{Net}_p(x>y) \]

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\[ \begin{array}{ccc}
    & b & \\
 a & 100 & 102 \\
 & 100 & c \\
\end{array} \]

Cop’d ignores size of margins in pairwise wins/losses **very indecisive**
1) Intro: Three voting rules

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\[ \begin{array}{c}
  a \\
  100 \\
  \text{b} \rightarrow \\
  102 \\
  \text{c} \\
  100 \\
  \text{a} \leftarrow \\
  100
\end{array} \]

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\[
\begin{array}{c}
\text{b} \\
\downarrow \\
100 \\
\uparrow \\
\text{a} \\
\downarrow \\
100 \\
\uparrow \\
\text{c} \\
\end{array}
\]

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This is **not** the standard definition of Borda Voting Rule

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**Scoring Rules**

- \( w = (w_1 \geq w_2 \geq \ldots \geq w_m) \) any vector of numerical **scoring weights**, \( w_1 > w_m \)

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- Standard Borda weights: \( m-1, m-2, \ldots, 0 \); evenly spaced \( 2, 1, 0 \) for \( m = 3 \)
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1) Intro: Three voting rules

Exercise 2

a) Scoring vectors \( w_1, \ldots, w_m \) and \( v_1, \ldots, v_m \) are **affinely equivalent** if there exist constants \( \gamma, \delta \) with \( \gamma > 0 \) and \( v_i = \gamma w_i + \delta \) for each \( i \). Show that
- affinely equivalent vectors induce same voting rule, and
- any two evenly spaced vectors are affinely equivalent.

b) Show symmetric Borda weights yield a total score = \( \beta(x) \).

**Scoring Rules**

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2) Social Choice Functions

Goal: select one alternative from a finite set $A$

1. Each voter (finitely many) casts a ballot

2. Apply some voting rule
2) Social Choice Functions

Goal: select one alternative from a finite set $A$

1. Each voter (finitely many) casts a ballot
2. Apply some voting rule

But ties are possible!
2) Social Choice Functions

Goal: select one alternative from a finite set A

1. Each voter (finitely many) casts a ballot
2. Apply some voting rule

Alternatives = . . . ?
- candidates for mayor of small town
- € budgets for new firehouse
- Estimates for amount of oil lying beneath a region
- (amend the constitution?) yes or no
- different versions of an immigration reform bill
- committees
2) Social Choice Functions

Goal: select one alternative from a finite set A

1. Each voter (finitely many) casts a ballot

2. Apply some voting rule

A ballot might be . . .
- an individual alternative
- a strict ranking of alternatives

Francine

```
  d
  a
  c
  b
  e
```

linear ordering \( \geq_F \) of \( A = \{a, b, c, d, e\} \)

\( \mathcal{L}(A) \) = the set of all possible linear orderings of \( A \).

\( |\mathcal{L}(A)| = m! \)
2) Social Choice Functions

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2. Apply some voting rule

A ballot might be . . .
- an individual alternative
- a strict ranking of alternatives
- a weak ranking of alternatives

Ahmed
d, e
c
a, b

$d \geq_A e$ and $e \geq_A d$ both hold, so we say “Ahmed is indifferent to $d$ and $e.”$ But maybe not . . .
2) Social Choice Functions

Goal: select one alternative from a finite set A
1. Each voter (finitely many) casts a ballot
2. Apply some voting rule

A ballot might be . . .
• an individual alternative
• a strict ranking of alternatives
• a weak ranking of alternatives
• yes or no or abstain or ...
• a set of 1 or more alternatives those you “approve” for mayor
• a separate score (1-10) assigned to each alternative
2) Social Choice Functions

Goal: select one alternative from a finite set A

1. Each voter (finitely many) casts a ballot

2. Apply some voting rule

A ballot might be . . .

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• a strict ranking of alternatives
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Goal: select one alternative from a finite set $A$

1. Each voter (finitely many) casts a ballot
2. Apply some voting rule

A ballot might be . . .
- an individual alternative
- a strict ranking of alternatives
- **a weak ranking of alternatives**
- yes or no or abstain or ... 
- a set of 1 or more alternatives *those you “approve” for mayor*
- a separate score (1-10) assigned to each alternative
There are many types of voting.

We focus on one type: Social Choice Functions *SCFs*
2) Social Choice Functions

- \( N = \{1, 2, \ldots, n\} \) set of \( n \) voters
- \( A = \) finite set of \( m \) alternatives
- \( C(A) = \{ X \subseteq A \mid X \neq \emptyset \} \)
- \( \geq_j = \) ballot cast by voter \( j \), an element of \( \mathcal{L}(A) \)
- \( P = (\geq_1, \geq_2, \ldots, \geq_n) \in \mathcal{L}(A)^n \) specifies a ballot for each voter \( j \in N \). \( P \) is a profile.
- A **SCF** is a function that assigns, to each election, one winner (or several, if a tie)
  \( f: \mathcal{L}(A)^n \rightarrow C(A) \)
- A SCF with no ties is **resolute**
- A **variable electorate** SCF handles profiles for all finite \( n \)

\[
\mathcal{L}(A)^{<\infty} = \bigcup \{ \mathcal{L}(A)^n \mid n \in \mathbb{N} \}
\]
\( f: \mathcal{L}(A)^{<\infty} \rightarrow C(A) \)
3) A taste of strategic manipulation

Consider profile $P_3$, in which Ali is one of the last 2 voters.

<p>| | | |</p>
<table>
<thead>
<tr>
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<tr>
<td>2</td>
<td>3</td>
<td>2</td>
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<td>e</td>
<td>d</td>
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Copeland scores are:

Cop(a) = 0
Cop(b) = -2
Cop(c) = 0
Cop(d) = 0
Cop(e) = 2
3) A taste of strategic manipulation

Consider profile $P_3$, in which Ali is one of the last 2 voters.

Copeland scores are:
- Cop(a) = 0
- Cop(b) = -2
- Cop(c) = 0
- Cop(d) = 0
- Cop(e) = 2*

$e$ loses to $d$ and to no one else
3) A taste of strategic manipulation

Consider profile $P_3$, in which $P_3 = \begin{array}{ccc} 2 & 3 & 2 \\ e & d & a \\ c & e & b \\ a & b & c \\ d & c & d \\ b & a & e \end{array}$

Ali is one of the last 2 voters.

Copeland scores are:

- $\text{Cop}(a) = 0$
- $\text{Cop}(b) = -2$
- $\text{Cop}(c) = 0$
- $\text{Cop}(d) = 0^*$
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*d loses to a, c by 4-3

$d$ beats $b$, $e$ by 5-2
3) A taste of strategic manipulation

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Ali is . . . unhappy!
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  $d$ beats $b, e$ by 5-2

Ali is . . . unhappy!

*Before* the election, Ali anticipates this bad outcome
3) A taste of strategic manipulation

Consider profile $P_3$, in which Ali is one of the last 2 voters.

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<table>
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<tbody>
<tr>
<td>$e$</td>
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Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. 
3) A taste of strategic manipulation

Consider profile $P_3$, in which Ali is one of the last 2 voters. Copeland scores are:

$\text{Cop}(a) = 0$

$\text{Cop}(b) = -2$

$\text{Cop}(c) = 0$

$\text{Cop}(d) = 0^*$

$\text{Cop}(e) = 2$

*d loses to a, c by 4-3*

*d beats b, e by 5-2*

Ali is . . . unhappy!

Before the election, Ali anticipates this bad outcome

Now, $P_3^*$

<table>
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Before the election, Ali anticipates this bad outcome

In $P_3^*$, $d$ now beats $a, c$ by 4-3
  $d$ beats $b$ by 6-1; $d$ beats $e$ by 4-3
3) A taste of strategic manipulation

Consider profile $P_3$, in which Ali is one of the last 2 voters.

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Ali is . . . unhappy!

**Before** the election, Ali anticipates this bad outcome

In $P_3$, $d$ now beats $a, c$ by 4-3

$d$ beats $b$ by 6-1; $d$ beats $e$ by 4-3

$d$ beats everyone else, winning the Copeland election.
3) A taste of strategic manipulation

- By misrepresenting her preferences Ali does better.

<table>
<thead>
<tr>
<th>P₃</th>
<th>2</th>
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Ali decides to misrepresent her preferences by reversing her ballot: e ≥ d ≥ c ≥ b ≥ a. Now, P₃*

In P₃*, d now beats a, c by 4-3

d beats b by 6-1; d beats e by 4-3

d beats everyone else, winning the Copeland election.
3) A taste of strategic manipulation

- By misrepresenting her preferences Ali does better – she has *manipulated* the election

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Ali decides to misrepresent her preferences by reversing her ballot: \( e \geq d \geq c \geq b \geq a \). Now, \( P_3^* \)

In \( P_3^* \), \( d \) now beats \( a, c \) by 4-3
\( d \) beats \( b \) by 6-1; \( d \) beats \( e \) by 4-3
\( d \) beats everyone else, winning the Copeland election.
3) A taste of strategic manipulation

- By misrepresenting her preferences Ali does better – she has *manipulated* the election
- How much better?

\[ P_3 \]

\[
\begin{array}{ccc}
2 & 3 & 2 \\
\hline
e & d & a \\
c & e & b \\
a & b & c \\
d & c & d \\
b & a & e \\
\end{array}
\]

Ali decides to misrepresent her preferences by reversing her ballot: \( e \geq d \geq c \geq b \geq a \). Now, \( P_3^* \)

In \( P_3^* \), \( d \) now beats \( a, c \) by 4-3
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3) A taste of strategic manipulation

- By misrepresenting her preferences Ali does better – she has **manipulated** the election
- How much better?
- We don’t know – cannot extract cardinal utilities from ordinal preferences.

\[ P_3 \]

\[ \begin{array}{ccc}
  e & d & a \\
  c & e & b \\
  a & b & c \\
  d & c & d \\
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\end{array} \]

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- By misrepresenting her preferences Ali does better – she has *manipulated* the election
- How much better?
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- In $P_3^*$, d beats each other alternative in the pairwise majority sense; d is a *Condorcet alternative*

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Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_3^*$

*In $P_3^*$, d now beats a, c by 4-3*

*d beats b by 6-1; d beats e by 4-3*

*d beats everyone else, winning the Copeland election. d is a Cond. Alt.*
3) A taste of strategic manipulation

• By misrepresenting her preferences Ali does better – she has **manipulated** the election

• How much better?

• We don’t know – cannot extract cardinal utilities from ordinal preferences.

• In $P_3^*$, d beats each other alternative in the pairwise majority sense; d is a **Condorcet alternative**

• Condorcet’s principle: if $x$ is a Condorcet alternative, it should win

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$P_3$

Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_3^*$

In $P_3^*$, d now beats a, c by 4-3

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- Condorcet’s principle: if x is a Condorcet alternative,* it should win *might not be any

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Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_3^*$

*In $P_3^*$, d now beats a, c by 4-3
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- By misrepresenting her preferences Ali does better – she has *manipulated* the election.
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- Condorcet’s principle: if x is a Condorcet alternative,* it should win *might not be any
- A SCF honoring this principle is called a *Condorcet extension*

\[
P_3
\begin{array}{ccc}
 2 & 3 & 2 \\
e & d & a \\
c & e & b \\
a & b & c \\
d & c & d \\
b & a & e \\
\end{array}
\]

Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_3^*$

\[In P_3^*, d now beats a, c by 4-3\]
\[d beats b by 6-1; d beats e by 4-3\]
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3) A taste of strategic manipulation

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Ali decides to misrepresent her preferences by reversing her ballot: e ≥ d ≥ c ≥ b ≥ a. Now, P₃*

In P₃*, d now beats a, c by 4-3

d beats b by 6-1; d beats e by 4-3

d beats everyone else, winning the Copeland election. **d is a Cond. Alt.**
3) A taste of strategic manipulation

- A SCF honoring this principle is called a **Condorcet extension**
- Copeland Rule is a Cond. Extn. (A Cond. Alt. uniquely gets the max poss. Copeland score m-1)

\[ P_3 \]

\[
\begin{array}{ccc}
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c & e & b \\
a & b & c \\
d & c & d \\
b & a & e \\
\end{array}
\]

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*In \( P_3^* \), \( d \) now beats \( a, c \) by 4-3

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- A SCF honoring this principle is called a **Condorcet extension**
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- **News Flash!** Borda, not Copeland, will be used for $P_3$

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$P_3$ matrix:

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Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_3^*$

In $P_3^*$, $d$ now beats $a$, $c$ by 4-3
$d$ beats $b$ by 6-1; $d$ beats $e$ by 4-3
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- A SCF honoring this principle is called a **Condorcet extension**
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- **News Flash!** Borda, not Copeland, will be used for P₃
- Who wins in P₃?

<table>
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Ali decides to misrepresent her preferences by reversing her ballot: e ≥ d ≥ c ≥ b ≥ a. Now, P₃*

In P₃*, d now beats a, c by 4-3

* d beats b by 6-1; d beats e by 4-3

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- **Who wins in P₃?** Still e. (6)ₚ₃

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<td>c</td>
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Ali decides to misrepresent her preferences by reversing her ballot: e ≥ d ≥ c ≥ b ≥ a. Now, P₃*

In P₃*, d now beats a, c by 4-3
d beats b by 6-1; d beats e by 4-3
D beats everyone else, winning the Copeland election. d is a Cond. Alt.
3) A taste of strategic manipulation

- A SCF honoring this principle is called a **Condorcet extension**
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- **News Flash!** Borda, not Copeland, will be used for \( P_3 \)
- Who wins in \( P_3 \)? Still e. \((6)_{sym}\)
- Who wins in \( P_3^* \)?

\[
\begin{array}{ccc}
2 & 3 & 2 \\
e & d & a \\
c & e & b \\
a & b & c \\
d & c & d \\
b & a & e \\
\end{array}
\]

Ali decides to misrepresent her preferences by reversing her ballot: \( e \geq d \geq c \geq b \geq a \). Now, \( P_3^* \)

- In \( P_3^* \), \( d \) now beats \( a, c \) by 4-3
- \( d \) beats \( b \) by 6-1; \( d \) beats \( e \) by 4-3
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- Who wins in P₃? Still e. \((6)_{\text{sym}}\)
- Who wins in P₃*? Still e. \((14)\)

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Ali decides to misrepresent her preferences by reversing her ballot: \(e \geq d \geq c \geq b \geq a\). Now, P₃*

In P₃*, d now beats a, c by 4-3

d beats b by 6-1; d beats e by 4-3

d beats everyone else, winning the Copeland election. **d is a Cond. Alt.**
3) A taste of strategic manipulation

- A SCF honoring this principle is called a **Condorcet extension**
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- **News Flash!** Borda, not Copeland, will be used for $P_3$
- Who wins in $P_3$? Still e. $(6)_{sym}$
- Who wins in $P_3^*$? Still e. $(14)$
- Is Borda a Condorcet Ext’n?

\[
\begin{array}{ccc}
\text{e} & \text{d} & \text{a} \\
\text{c} & \text{e} & \text{b} \\
\text{a} & \text{b} & \text{c} \\
\text{d} & \text{c} & \text{d} \\
\text{b} & \text{a} & \text{e} \\
\end{array}
\]

Ali decides to misrepresent her preferences by reversing her ballot: \( e \geq d \geq c \geq b \geq a \). Now, $P_3^*$

In $P_3^*$, \( d \) now beats \( a, c \) by 4-3

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**News Flash!** Borda, not Copeland, will be used for $P_3$

**Who wins in $P_3$?** Still e. (6)$_{sym}$

**Who wins in $P_3$?** Still e. (14)

**Is Borda a Condorcet Extn’?** no!

**EXERCISE** 3 Show Borda can never be manipulated via reversal

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<tr>
<th>Borda</th>
<th>Copeland</th>
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Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_{3^*}$

In $P_{3^*}$, d now beats a, c by 4-3

d beats b by 6-1; d beats e by 4-3

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3) A taste of strategic manipulation

- A SCF honoring this principle is called a **Condorcet extension**
- Copeland Rule is a Cond. Extn. (A Cond. Alt. uniquely gets the max poss. Copeland score m-1)

**News Flash!** Borda, not Copeland, will be used for $P_3$
- Who wins in $P_3$? Still e. ($6)_{sym}$
- Who wins in $P_3^*$? Still e. ($14$)

**Is Borda a Condorcet Ext’n?** no!

**EXERCISE 3** Show Borda can never be manipulated via reversal

- In $P_3$, can Ali manip’te Borda?

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In $P_3^*$, $d$ now beats $a, c$ by 4-3
- $d$ beats $b$ by 6-1; $d$ beats $e$ by 4-3
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- **News Flash!** Borda, not Copeland, will be used for $P_3$
- Who wins in $P_3$? Still $e$. ($6)_{sym}$
- Who wins in $P_3^*$? Still $e$. ($14$)
- Is Borda a Condorcet Ext’n? **no!**

**EXERCISE** 3 Show Borda can never be manipulated via reversal

- In $P_3$, can Ali manip’te Borda?
- Yes: lift $d$ to top, push others down. $e: (6)$  $d: (4)\rightarrow (10)$

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$P_3$

Ali decides to misrepresent her preferences by reversing her ballot: $e \geq d \geq c \geq b \geq a$. Now, $P_3^*$

In $P_3^*$, $d$ now beats $a$, $c$ by 4-3

$d$ beats $b$ by 6-1; $d$ beats $e$ by 4-3

$d$ beats everyone else, winning the Copeland election. $d$ is a Cond. Alt.
3) A taste of strategic manipulation

**Definition**  An SCF \( f \) is *single-voter manipulable* if \( \exists \) profiles \( P, P^* \) and voter \( v \) s.t. \( f(P^*) >_v f(P) \), where \( P^* \) is obtained from \( P \) by having \( v \) alone switch ballots from \( \geq_v \) to \( \geq_{v^*} \), *with no ties in \( f(P) \) or \( f(P^*) \)*.
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**Interpretation**
- $v$’s ballot $\geq_v$ in $P$ = his sincere ranking
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**Definition**  An SCF $f$ is *single-voter manipulable* if $\exists$ profiles $P$, $P^*$ and voter $v$ s.t. $f(P^*) \succ_v f(P)$, where $P^*$ is obtained from $P$ by having $v$ alone switch ballots from $\succeq_v$ to $\succeq_v^*$, *with no ties in* $f(P)$ *or* $f(P^*)$.

**Interpretation**
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- $v$’s ballot $\geq_v$ in $P$ = his sincere ranking
- $v$’s ballot $\geq_{v^*}$ in $P^*$ = an insincere ranking (manip. attempt)
- $f(P^*) >_v f(P)$? the attempt succeeds: according to his sincere ranking $\geq_v$, he strictly prefers outcome from insincere ballot
3) A taste of strategic manipulation

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**Definition** An SCF f is *single-voter manipulable* if ∃ profiles P, P* and voter v s.t. f(P*) >_v f(P), where P* is obtained from P by having v alone switch ballots from ≥_v to ≥_v*, with no ties in f(P) or f(P*)

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- v’s ballot ≥_v in P = his sincere ranking
- v’s ballot ≥_v* in P* = an insincere ranking (manip. attempt)
- f(P*) >_v f(P)? the attempt succeeds

*What goes wrong with ties?*
- Say a ≥_v b ≥_v c
- f(P) = {a,c}, f(P*) = {b}
- Does he prefer b alone to an a-c tie? It depends!
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*What goes wrong with ties?*

*If a rule always yields ties, it is never s-v manipulable*

- Say \( a \geq_v b \geq_v c \)
- \( f(P) = \{a,c\}, f(P^*) = \{b\} \)
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4) The GST: Gibbard-Satterthwaite Theorem and Arrow’s Impossibility Theorem
4a) The GST: Gibbard-Satterthwaite Theorem

**Theorem** (Alan Gibbard, Mark Satterthwaite)
Let f be any SCF for three or more alternatives.

**If** f is:

- resolute
- nonimposed
- and strategyproof

**then** f must be a dictatorship
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**then** f must be a dictatorship – winner is dictator’s top-ranked alternative
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4b) Arrow’s Impossibility Theorem

**Theorem** (Kenneth Arrow) Let \( f \) be any Social Welfare Function (SWF) for **three or more alternatives**.

**Social Welfare Function**
- Ballots are linear orders of \( A \) (as before), . . .
- but the outcome \( F(P) \) of an election is a weak order of \( A \)
4b) Arrow’s Impossibility Theorem

**Theorem** (Kenneth Arrow) Let f be any Social Welfare Function (SWF) for three or more alternatives.

If f satisfies:

- the weak Pareto property for SWFs
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A SWF $F$ satisfies the Weak Pareto property if

- whenever each voter $i$ ranks $x >_i y$,
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**Social Welfare Function**

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- but the outcome \( F(P) \) of an election is a weak order of \( A \)

A SWF \( F \) satisfies the **Weak Pareto property** if

- whenever each voter \( i \) ranks \( x >_i y \), \( F \) respects unanimity in strict preferences
- the outcome under \( F \) has \( x > y \)
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A SWF F satisfies **IIA** if

- for each pair x, y of alternatives
- the relative ranking of x VS y in the outcome F(P)
- depends only on the relative ranking of x VS y in the ballots
4b) Arrow’s Impossibility Theorem

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P $\mapsto$ P*: no individual voter changes **relative** ranking of x VS y.
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P $\mapsto$ P*: no individual voter changes **relative** ranking of x VS y. So IIA says “If F(P) = x > y > a > b > c then F(P*) must have x > y”
4b) Arrow’s **Impossibility** Theorem

**Theorem** (Kenneth Arrow) Let $f$ be any Social Welfare Function (SWF) for three or more alternatives.

If $f$ satisfies:

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Equivalently . . . **No** SWF for 3 or more alternatives satisfies weak Pareto + \( \text{IIA} \) + nondictatoriality
5) Axioms I  Pareto property, anonymity, neutrality

- A profile of 201 voters

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- For the moment, only top choices visible
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- Based on this limited info, b wins
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- Argument for b is stronger than “b is the plurality winner” . . .
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WHY?
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WHY?

Majoritarian Principle
5) Axioms I  Pareto property, anonymity, neutrality

• A profile of 201 voters

• For the moment, only top choices visible

• Based on this limited info, \textcolor{orange}{b} wins

• We see the hidden info

\begin{align*}
101 & & 100 \\
\text{b} & & \text{a} \\
\text{a} & & \text{d} \\
\text{c} & & \text{e} \\
\text{d} & & \text{f} \\
\text{e} & & \text{b} \\
\text{f} & & \text{c}
\end{align*}
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- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, \( b \) wins
- We see the hidden info
- Should \( b \) still win?

\[
\begin{array}{c|c|c}
101 & 100 \\
\hline
b & a \\
\hline
a & d \\
\hline
c & e \\
\hline
d & f \\
\hline
e & b \\
\hline
f & c \\
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- For the moment, only top choices visible
- Based on this limited info, \textbf{b} wins
- We see the hidden info
- Should \textbf{b} still win?
- Or should it be \textbf{a}?
5) Axioms I  Pareto property, anonymity, neutrality

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- Based on this limited info, **b** wins
- We see the hidden info
- Should **b** still win? **hands**
- Or should it be **a**?

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5) Axioms I  Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, $b$ wins
- We see the hidden info
- Should $b$ still win? **hands**
- Or should it be $a$? **hands**
5) Axioms I  Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, b wins
- We see the hidden info
- Should b still win? hands
- Or should it be a? hands

b is Condorcet alt, so b wins
Copeland
5) Axioms I  Pareto property, anonymity, neutrality

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Copeland

a is Borda winner. . .  Borda
“favors compromise over Maj.”
5) Axioms I  Pareto property, anonymity, neutrality

- A profile of 201 voters
- For the moment, only top choices visible
- Based on this limited info, \( b \) wins
- We see the hidden info
- Should \( b \) still win? hands
- Or should it be \( a \)? hands

\( b \) is Condorcet alt, so \( b \) wins
\( a \) is Borda winner. . . Borda
“favors compromise over Maj.”
5) Axioms I  Pareto property, anonymity, neutrality

• A profile of 201 voters

• Someone says it is c, not a or b, who should win
5) Axioms I  Pareto property, anonymity, neutrality

• A profile of 201 voters

• Someone says it is c, not a or b, who should win

• Counterargument is ... ?
5) Axioms I  Pareto property, anonymity, neutrality

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5) Axioms I  Pareto property, anonymity, neutrality

- A profile of 201 voters
- Someone says it is c, not a or b, who should win?
- Counterargument is ...
- Every voter prefers b to c

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- Someone says it is c, not a or b, who should win
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- c is “Pareto dominated” by b
5) Axioms I  Pareto property, anonymity, neutrality

- A profile of 201 voters
- Someone says it is c, not a or b, who should win
- Counterargument is ... ?
- Every voter prefers b to c
- c is “Pareto dominated” by b

**Axiom** An SCF f satisfies the **Pareto Principle** if f(P) never includes a Pareto dominated alternative
5) Axioms I  Pareto property, anonymity, neutrality

Easy Theorem: *Pareto Principle* is satisfied by
- Plurality rule
- Borda
- Copeland

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5) Axioms I  Pareto property, anonymity, neutrality

**Easy Theorem: Pareto Principle** is satisfied by

- Plurality rule
- Borda
- Copeland

(And by most “reasonable” SCFs)

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5) Axioms I  Pareto property, anonymity, neutrality

Easy Theorem: *Pareto Principle* is satisfied by

- Plurality rule
- Borda
- Copeland

(And by most “reasonable” SCFs)

Pareto implies that winner for this 201-vote profile is *a* or *b*
5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is a.

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5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is a.

- Anna is a type I voter (among 1st group of 101);
  Stevo is type II

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5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is a.

- Anna is a type I voter (among 1st group of 101);
- Stevo is type II
- Anna is convinced by Sara to change her ballot to type II.
5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is a.

- Anna is a type I voter (among 1st group of 101);
  Stevo is type II
- Anna is convinced by Sara to change her ballot to type II.
- At same time Stevo is convinced to change his ballot to type I
5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is a.

- Anna is a type I voter (among 1st group of 101);
- Stevo is type II
- Anna is convinced by Sara to change her ballot to type II.
- At same time Stevo is convinced to change his ballot to type I

After both switches, how should outcome change?
5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is **a**.

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- Anna is a type **I** voter (among 1st group of 101);
- Stevo is type **II**

- Anna is convinced by Sara to change her ballot to type **II**.

- At same time Stevo is convinced to change his ballot to type **I**

After both switches, how should outcome change? It depends!
5) Axioms I  Pareto property, anonymity, neutrality

Assume the winner for this profile is a.

• Anna is a type I voter (among 1st group of 101);
  Stevo is type II

• Anna is convinced by Sara to change her ballot to type II.

• At same time Stevo is convinced to change his ballot to type I

After both switches, how should outcome change?
It depends! In some contexts, not at all.
5) Axioms I Pareto property, anonymity, neutrality

**Axiom** An SCF \( f \) is *anonymous* if each pair of voters play interchangeable roles: 
\[ f(P) = f(P^*) \]
whenever \( P^* \) is obtained from \( P \) by swapping ballots of 2 voters.

After both switches, how should outcome change? It depends! In some contexts, *not at all.*
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Axiom An SCF \( f \) is **anonymous** if each pair of voters play interchangeable roles:
\[ f(P) = f(P^*) \text{ whenever } P^* \text{ is obtained from } P \text{ by swapping ballots of } 2 \text{ voters.} \]

Math This says \( f(P) = f(\tau P) \) for each **transposition** \( \tau \) of voters.

Transpositions generate the full symmetric group. So \( f(P) = f(\sigma P) \) for each **permutation** \( \sigma \) of the set \( N \) of voters.
5) Axioms I  Pareto property, anonymity, neutrality

**Axiom** An SCF $f$ is *anonymous* if each pair of voters play interchangeable roles: $f(P) = f(P^*)$ whenever $P^*$ is obtained from $P$ by swapping ballots of 2 voters.

Anonymity is a very *strong* form of equal influence by voters. Non-dictatoriality is a very *weak* form.

**Math** This says $f(P) = f(\tau P)$ for each *transposition* $\tau$ of voters.

Transpositions generate the full symmetric group. So $f(P) = f(\sigma P)$ for each *permutation* $\sigma$ of the set $N$ of voters.
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Again, assume the winner for this profile is a.

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5) Axioms I  Pareto property, anonymity, neutrality

Again, assume the winner for this profile is a.

• This time, switch candidate a with candidate f (in all ballots)
5) Axioms I  Pareto property, anonymity, neutrality

Again, assume the winner for this profile is \textbf{a}.

- This time, switch \textit{candidate} \textbf{a} with \textit{candidate} \textbf{f} (in all ballots)
5) Axioms I  Pareto property, anonymity, neutrality

Again, assume the winner for this profile is \textbf{a}.

- This time, switch \textit{candidate} \textbf{a} with \textit{candidate} \textbf{f} (in all ballots)

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101 & 100 \\
b & f \\
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5) Axioms I    Pareto property, anonymity, neutrality

Again, assume the winner for this profile is \textbf{a}.

- This time, switch \textit{candidate} a with \textit{candidate} f (in all ballots)
- After the switches, how should outcome change?
- Assume the voting rule treats candidates equivalently.

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Again, assume the winner for this profile is **a**.

- This time, switch *candidate* a with *candidate* f (in all ballots)
- After the switches, how should outcome change?
- Assume the voting rule treats candidates equivalently.
- **f** should win, post switch
5) Axioms I  Pareto property, anonymity, neutrality

Again, assume the winner for this profile is \textbf{a}.

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\hspace{1cm} \textbf{Axiom} An SCF f is \textit{neutral} if each pair of candidates play interchangeable roles.

- \textbf{f} should win, post switch
5) Axioms I  Pareto property, anonymity, neutrality

Again, assume the winner for this profile is $a$.

- This time, switch *candidate* $a$ with *candidate* $f$ (in all ballots)
- After the switches, how should outcome change?
- Assume the voting rule treats candidates equivalently.
- $f$ should win, post switch

**Axiom** An SCF $f$ is *neutral* if each pair of candidates play interchangeable roles:  
$$f(P^\tau) = \tau[f(P)]$$

whenever $P^\tau$ is obtained from $P$ by swapping 2 alternatives in all ballots.
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**Axiom** An SCF \( f \) is **neutral** if each pair of candidates play interchangeable roles: \( f(P^{\tau}) = \tau[f(P)] \) whenever \( P^{\tau} \) is obtained from \( P \) by swapping 2 alternatives in all ballots.

Again, we can replace \( \tau \) with \( \sigma \): \( f(P^\sigma) = \sigma^{-1}[f(P)] \)

Why use *inverse* of \( \sigma \)?
5) Axioms I  Pareto property, anonymity, neutrality

• These three axioms are easy to satisfy: many rules satisfy all of them
5) Axioms I Pareto property, anonymity, neutrality

- These three axioms are easy to satisfy: many rules satisfy all of them
- But they already show
  
  *you can’t always get what you want*
5) Axioms I  Pareto property, anonymity, neutrality

• These three axioms are easy to satisfy: many rules satisfy all of them

• But they already show

you can’t always get what you want  And certainly, the GST and Arrow’s Theorem show this
5) Axioms I  Pareto property, anonymity, neutrality

• These three axioms are easy to satisfy: many rules satisfy all of them

• But they already show *you can’t always get what you want*

• Together, they have negative implications for resoluteness
5) Axioms I  Pareto property, anonymity, neutrality

• These three axioms are easy to satisfy: many rules satisfy all of them

• But they already show you can’t always get what you want

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• A profile for 3k voters, m alternatives
5) **Axioms I** Pareto property, anonymity, neutrality

- These three axioms are easy to satisfy: many rules satisfy all of them.
- But they already show *you can’t always get what you want*
- Together, they have negative implications for resoluteness
- A profile for 3k voters, m alternatives

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5) Axioms I  Pareto property, anonymity, neutrality

• These three axioms are easy to satisfy: many rules satisfy all of them

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  a & c & b \\
  b & a & c \\
  c & b & a \\
  x_1 & x_1 & x_1 \\
  \vdots & \vdots & \vdots \\
  x_{m-2} & x_{m-2} & x_{m-2}
  \end{array}
  \]

• But they already show *you can’t always get what you want*

• Together, they have negative implications for resoluteness

• A profile for 3k voters, m alternatives

  We’ll show a 3-way tie is forced
5) **Axioms I**  Pareto property, anonymity, neutrality

- Pareto $\implies f(P) \subseteq \{a,b,c\}$

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We’ll show a 3-way tie is forced
5) Axioms I  Pareto property, anonymity, neutrality

- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$
  
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- WLOG assume $a \in f(P)$

We’ll show a 3-way tie is forced
5) Axioms I  Pareto property, anonymity, neutrality

- Pareto $\Rightarrow f(P) \subseteq \{a, b, c\}$
- WLOG assume $a \in f(P)$
- First, permute voters

\[
\begin{array}{ccc}
  k & k & k \\
a & c & b \\
b & a & c \\
c & b & a \\
x_1 & x_1 & x_1 \\
\vdots & \vdots & \vdots \\
x_{m-2} & x_{m-2} & x_{m-2}
\end{array}
\]

We’ll show a 3-way tie is forced
5) Axioms I  Pareto property, anonymity, neutrality

- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$  
  \[k\quad k\quad k\]

- WLOG assume $a \in f(P)$  
  \[a\quad c\quad b\]

- First, permute voters  
  \[b\quad a\quad c\]

- $\rho$: 1st $k \rightarrow$ last $k \rightarrow$ mid $k$  
  \[c\quad b\quad a\]

\[x_1\quad x_1\quad x_1\]
\[\vdots\quad \vdots\quad \vdots\]
\[x_{m-2}\quad x_{m-2}\quad x_{m-2}\]

We’ll show a 3-way tie is forced
5) Axioms I  Pareto property, anonymity, neutrality

- Pareto ⇒ \( f(P) \subseteq \{a,b,c\} \)

- WLOG assume \( a \in f(P) \)

- First, permute voters

- \( \rho: 1^{st} k \rightarrow \text{last } k \rightarrow \text{mid } k \)

We’ll show a 3-way tie is forced
5) Axioms I  Pareto property, anonymity, neutrality

- Pareto $\Rightarrow f(P) \subseteq \{a,b,c\}$
- WLOG assume $a \in f(P)$
- First, permute voters
- $\rho$: 1$^{st}$ $k$ $\rightarrow$ last $k$ $\rightarrow$ mid $k$
- $f(\rho P) = f(P)$

We’ll show a 3-way tie is forced
5) Axioms I  Pareto property, anonymity, neutrality

- Pareto \( \Rightarrow f(P) \subseteq \{a,b,c\} \)
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  \[ \begin{array}{ccc} \ a & b & c \end{array} \]

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  \[ \begin{array}{ccc} \ x_1 & x_1 & x_1 \end{array} \]

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  \[ \begin{array}{ccc} \ : & : & : \end{array} \]

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  \[ \begin{array}{ccc} \ x_{m-2} & x_{m-2} & x_{m-2} \end{array} \]

• f(P) = f((ρP)^{⊙})  
  = σ^{-1}[f(ρP)]  

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• $f(P) = f((\rho P)^{\circ})$
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- So $f(P)$ is closed under $\sigma^{-1}$, so $\{a, b, c\} \subseteq f(P)$

We’ll show a 3-way tie is forced.
5) Axioms I  Pareto property, anonymity, neutrality

Assume 5k voters and 5 ≤ m = # alt’s.

We’ll show a 3-way tie is forced
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**k-approval:**

\( w = (1, \ldots, 1, 1, 0, \ldots, 0, 0) \)

with \( k \) 1s
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II Condorcet Extensions

Recall: A Condorcet alternative $a$ satisfies $a >^\mu b$ for each alternative $b \neq a$
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1. Can more than one alternative a have $SS(a) > 0$?
2. Suppose $SS(a) > 0 \ldots$ what can you say about alt. a?
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Top Cycle: A subset \( X \subseteq A \) is a dominating set if \( x >^\mu y \) holds for each \( x \in X, y \not\in X \).
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**Top Cycle:** A subset \(X \subseteq A\) is a **dominating set** if \(x >^\mu y\) holds for each \(x \in X, y \notin X\)

\(TC(P) = \) the smallest dominating set (which is unique)
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$TC(P) =$ the smallest dominating set (which is unique)

Why is Top Cycle a Condorcet Extension?
Exercises

• This section contains precise versions of problems mentioned on slides
• Only do the ones you find interesting (there are too many for you to do all right now)
• Most of the tutorial is based on Chapter 2 of the *Handbook of Computational Social Choice*, Cambridge University Press, 2016. You may find the chapter helpful for these problems.
• Free PDF of the book at http://www.cambridge.org/download_file/898428
• To open the PDF use password: cam1CSC
1) Copeland scoring

• Recall **symmetric Copeland score** is given by
  \[ \text{Cop}(x) = | \{ y | x >^\mu y \} | - | \{ y | y >^\mu x \} | \]

• **Asymmetric Copeland score** is given by
  \[ \text{Cop}^{\text{Ass.}}(x) = | \{ y | x >^\mu y \} | \]

• **Asymmetric+ Copeland score** is given by
  \[ \text{Cop}^{\text{Ass.}+}(x) = | \{ y | x >^\mu y \} | + \left( \frac{1}{2} \right) | \{ y | y =^\mu x \} | \]

Are these three rules all the same? All different? Answer as completely as possible.

*We write \( y =^\mu x \) when \( \text{Net}_p(x>y) = 0 \). You will need to consider profiles for an even number of voters, making \( y =^\mu x \) possible.*
Exercises

2) Scoring weights and affine equivalence

• Scoring vectors $w = w_1, \ldots, w_m$ and $v = v_1, \ldots, v_m$ are **affinely equivalent** if there exist constants $\gamma, \delta$ with $\gamma > 0$ such that $v_i = \gamma w_i + \delta$ for each $i$.

• Prove that two scoring vectors $w, v$ induce the same scoring rule iff they are affinely equivalent.

• Prove that any two evenly spaced vectors are affinely equivalent.

• Prove that **symmetric** Borda weights $m-1, m-3, \ldots, -m+1$ yield a total score of $\beta(x)$ for each alternative $x$.

  \[
  Recall \ that \ \beta(x) = \sum_{y \in A} Net_p(x > y)
  \]
Exercises

3) Reversal Manipulation We saw Copeland can be manipulated via reversal: a profile P exists for which some voter i can, by completely reversing her ranking, switch the winning alternative from x to some alternative y whom she sincerely prefers (she ranked y over x before reversing)

• Prove that Borda cannot be manipulated via reversal (the same argument shows all scoring rules are similarly immune)

• Prove that Simpson-Kramer can be manipulated via reversal

• Difficult: Prove that every resolute Condorcet extension for 4 or more alternatives can be manipulated via reversal