# Classification of All Condorcet Domain Structures for Four Alternatives

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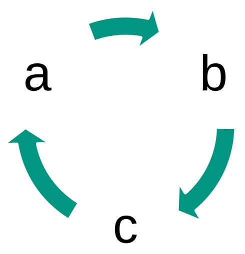
## Pairwise majority voting

Pairwise majority voting is one of the most intuitive methods with many desirable properties, but it has one major defect: the possibility of cyclical outcomes.

Voter 1 : a > b > c
Voter 2 : c > a > b →
Voter 3 : b > c > a




One way out of this problem are domain restrictions, which guarantee the absence of cyclical outcomes: Condorcet domains (CDs).

## Literature on Condorcet domain classes

There are several classes of Condorcet domains known in the literature:

- single-peaked preferences (Black 1948, Arrow 1951)
- value-restriction (Sen 1966)
- order-restriction (Rothstein 1990), single-crossing preferences (Gans and Smart 1996)
- intermediate preferences on a median graph (Demange 2012, Puppe and Slinko 2015)

There is an active research field which targets the search of large CDs (Monjardet 2009):

| n | g(n) | f(n) | AS(n) | RS(n) |
|---|---|---|---|---|
| 3 | 4 | 4 | 4 | 4 |
| 4 | 9 | 9 | 9 | 8 |
| 5 | 20 | 20 | 20 | 16 |
| 6 | 45 | 45 | 45 | 36 |
| 7 | 100 | ? | 100 | 81 |
| 8 | ? | ? | 222 | 180 |

g(n): maximum size of connected CDs
f (n): maximum size of CDs
AS(n) / RS(n): size of CDs using alternating scheme / replacement scheme (Fishburn 1997)

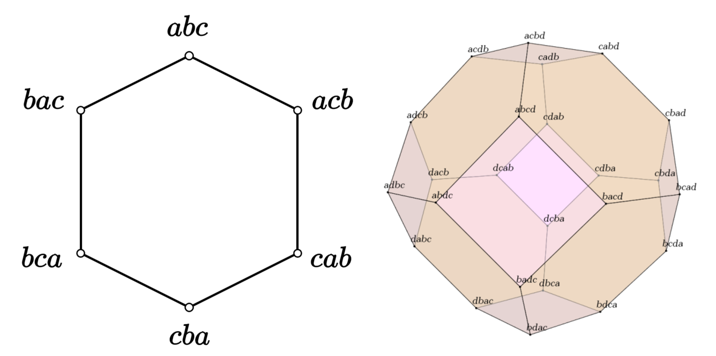
## Contribution

We develop a computer program to determine all maximal Condorcet domains for strict and weak preference orders using the median graph characterization of Condorcet domains by Puppe and Slinko (2015). For four alternatives we identify, describe and visualize 18 Condorcet domain structures for strict preferences. To the best of our knowledge, this is the first work to provide a complete overview and description of all Condorcet domains (for small numbers of alternatives). For weak preferences we identify over 50 connected Condorcet domain structures.

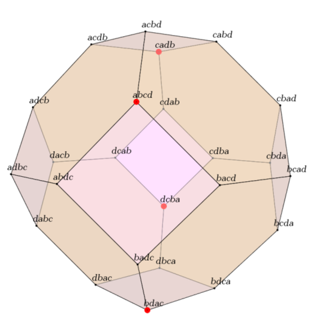
Calculate graph *perm* (graph of P(A)) → Creates all permutations of a given set A of alternatives
Choose arbitrary starting vertex $v_0$
Create graph *medGraph*($v_0$) → Creates the initial median graph with one element
Create set *residuals*(perm - $v_0$) → Creates the initial set of residuals (all elements except of $v_0$)

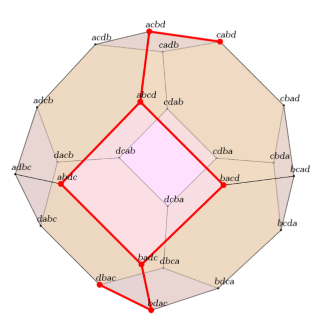
**procedure** CDRECURSIVE(medGraph, residual)
  **for** all elements *v* in *residual* **do**
    **if** (*medGraph* + *v*) is not median graph **then** remove *v* from *residual*
    **end if**
  **end for**
  **if** *residual* is empty **then** save *medGraph* to *file*
  **else**
    **for** all elements *w* in *residual* **do**
      CDRECURSIVE(*medGraph* + *w*, *residual* – *w*)
      delete *w* from *residual*
    **end for**
  **end if**
**end procedure**
**for** all domains *d* in *file* **do**
  check whether *d* is maximal
**end for**

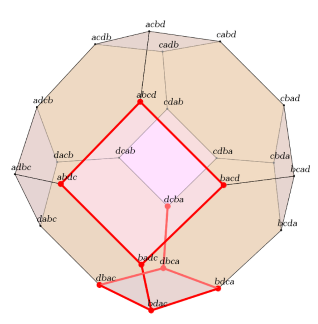
Graphs of P(A) for m=3 and m=4:

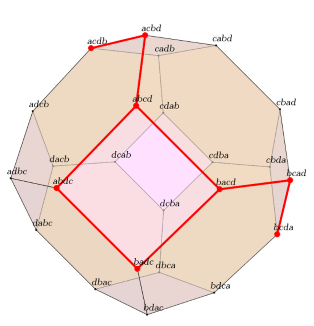


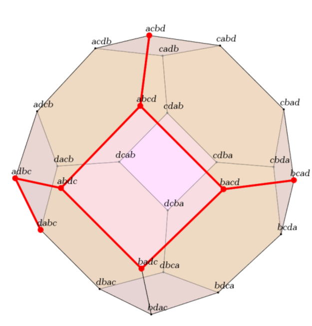

## Selected Condorcet domain structures

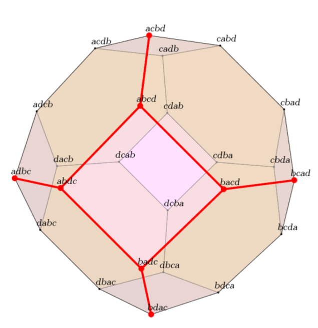

The four unconnected

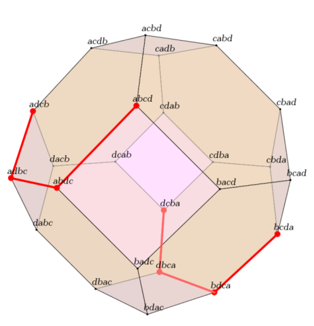

The single-peaked

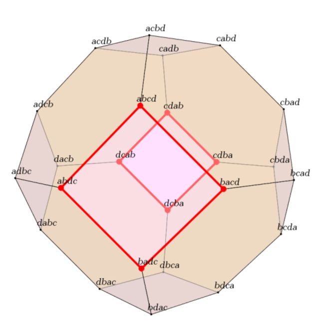

The maximum one

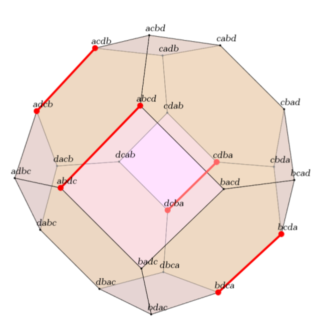

The crab


The half-crab-half-sun


The sun


The unconnected lines


The unconnected squares


Four unconnected pairs

| Class | # nodes | maximal width | minimally rich | connected | polarity | domain product | tiling type | single-peaked | single-crossing | # isomor-phism |
|---|---|---|---|---|---|---|---|---|---|---|
| The four unconn | 4 | y | y* | n | n | n | n | n | n | 6 |
| The snake | 7 | y | n | y | n | n | y | n | y | 48 |
| The broken snake | 7 | y | y | n | n | n | n | n | n | 48 |
| The single-peaked | 8 | y | y | y | n | n | y | y | n | 24 |
| The crab | 8 | n | y | y | n | n | n | n | n | 24 |
| The sun | 8 | n | n | y | n | y | n | n | n | 6 |
| The h-crab-h-sun | 8 | n | n | y | n | n | n | n | n | 48 |
| The unconn sq | 8 | y | y* | n | n | y | n | n | n | 3 |
| The unconn lines | 8 | y | y | n | y | n | n | n | n | 24 |
| Four unconn pairs | 8 | y | y* | n | y | y | n | n | n | 12 |
| Boring I | 8 | y | y | n | n | n | n | n | n | 48 |
| Boring II | 8 | y | n | n | n | n | n | n | n | 48 |
| Boring III | 8 | y | n | n | n | n | n | n | n | 48 |
| Boring IV | 8 | n | y | n | n | n | n | n | n | 24 |
| Boring V | 8 | y | y | n | n | n | n | n | n | 24 |
| Boring VI | 8 | y | y | n | n | n | n | n | n | 12 |
| Boring VII | 8 | n | n | n | n | n | n | n | n | 24 |
| The maximum one | 9 | y | n | y | n | n | y | n | n | 24 |